

Plan: Local existence and uniqueness of solutions of

$$\mathbf{X}' = F(\mathbf{X}, t)$$

$$\mathbf{X}(t_0) = \mathbf{X}_0$$

Results from analysis

For $a > 0$ & $I = [t_0 - a, t_0 + a]$,

the space

$$C(I, \mathbb{R}^n) = \{ \mathbf{X} \rightarrow \mathbb{R}^n \mid x_1, \dots, x_n \in C(I) \}$$

with the norm

$$\| \mathbf{X} \|_{\infty} := \max_{t \in I} | \mathbf{X}(t) | \quad \text{is a}$$

Banach space.

Importantly, if $\{ \mathbf{X}_n \}_n$ is Cauchy,

then $\lim_{n \rightarrow \infty} \mathbf{X}_n =: \mathbf{X} \in C(I, \mathbb{R}^n)$.

Def 1 For a domain $D \subset \mathbb{R}^n$,

$$C^1(D, \mathbb{R}^n) = \{F: D \rightarrow \mathbb{R}^n \mid f_i, \partial_{x_j} f_i \in C(D, \mathbb{R}) \mid 1 \leq i, j \leq n\}$$

Def 2 $F \in C^1(D, \mathbb{R}^n)$ is called locally Lipschitz if for each $x \in D$, $\exists \delta > 0$ and $L > 0$ s.t.

$$|F(x) - F(z)| \leq L|x - z| \quad \forall x, z \in \overline{B}_\delta(x) \quad (1)$$

If there exists an $L > 0$ s.t. (1) holds for all $x \in D$, then F is called globally Lipschitz.

Lemma 1 $F \in C^1(D, \mathbb{R}^n) \Rightarrow F$ is locally Lipschitz.

Proof Let $DF(x) := \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$

(be the Jacobian of F). For a point $\bar{x} \in D$,
 $\exists \delta > 0$ s.t. $\bar{B}_\delta(\bar{x}) \subset D$.

Let $L := \max_{Y \in \bar{B}_\delta(\bar{x})} \|DF(Y)\|_2$

$$\leq \max_{Y \in \bar{B}_\delta(\bar{x})} \underbrace{\left(\sum_{i,j=1}^n \left(\frac{\partial f_i(Y)}{\partial x_j} \right)^2 \right)^{1/2}}_{\text{continuous in } Y} < \infty,$$

since $\frac{\partial f_i}{\partial x_j} \in C(\bar{B}_\delta(\bar{x}))$ & $\bar{B}_\delta(\bar{x})$ compact,
 and $\|A\|_2 \leq \|A\|_{\text{Frobenius}} := \sqrt{\sum_{i,j=1}^n a_{ij}^2}$ $\forall A \in \mathbb{R}^{n \times n}$.

For any $Y, Z \in \bar{B}_\delta(\bar{x})$, the line $U(s) = Z + s(Y - Z)$
 for $s \in [0, 1]$ satisfies $U([0, 1]) \subset \bar{B}_\delta(\bar{x})$,
 since the ball is a convex set



Hence $F(Y) - F(Z) = \int_0^1 \frac{d}{ds} F(U(s)) ds$
 $= \int_0^1 DF(U(s)) \underbrace{U'(s)}_{(Y-Z)} ds$

$$\begin{aligned}
 \Rightarrow |F(Y) - F(Z)| &\leq \int_0^1 |DF(U(s))(Y-Z)| ds \\
 &\leq \int_0^1 \underbrace{\|DF(U(s))\|_2}_{\leq L} |Y-Z| ds \\
 &\leq L |Y-Z|
 \end{aligned}$$



P-L thm for autonomous setting.

Theorem 1 Let $F \in C^1(D)$ and consider

$$\dot{X} = F(X)$$

$$X(t_0) = X_0 \in D.$$

Then there exists a unique local solution $X \in C^1(I, \mathbb{R}^n)$, where

$I := [t_0 - a, t_0 + a]$ for some $a > 0$.

Proof Existence:

Pick a $\delta > 0$ s.t. F satisfies (1) on $\overline{B}_\delta(\mathbb{X}_0)$ for an $L > 0$ and let $M := \max_{\mathbb{Z} \in \overline{B}_\delta(\mathbb{X}_0)} |F(\mathbb{Z})|$. Set $a = \min\left(\frac{\delta}{M}, \frac{1}{2L}\right)$.

Consider sequence

$$\left. \begin{aligned} \mathbb{X}_1(t) &= \mathbb{X}_0 + \int_{t_0}^t F(\mathbb{X}_0) ds \\ \mathbb{X}_{k+1}(t) &= \mathbb{X}_0 + \int_{t_0}^t F(\mathbb{X}_k(s)) ds \end{aligned} \right\} \begin{array}{l} t \in I \\ k = 1, 2, \dots \end{array}$$

Claim $\mathbb{X}_k \in C(I, \overline{B}_\delta(\mathbb{X}_0))$ $k = 1, 2, \dots$

Pf: $|\mathbb{X}_1(t) - \mathbb{X}_0| \leq |F(\mathbb{X}_0)| |t - t_0|$
 $\leq Ma \leq \delta \quad \forall t \in I.$

Suppose $\Sigma_k \in C(I, \bar{B}_\delta(\Sigma_0))$ for some $k \in \mathbb{N}$
(holds for $k=1$), then

$$|\Sigma_{k+1}(t) - \Sigma_0| \leq \left| \int_{t_0}^t \underbrace{|F(\Sigma_k(s))|}_{\leq M} ds \right| \quad (\Sigma_k \in \bar{B}_\delta(\Sigma_0))$$

$$\leq Ma \leq \delta \quad \forall t \in I$$

Claim 1 follows by induction

Claim 2. $\{\Sigma_k\} \subset C(I, \bar{B}_\delta(\Sigma_0)) \subset C(I, \mathbb{R}^n)$
is a Cauchy sequence on $(C(I, \mathbb{R}^n), \|\cdot\|_\infty)$

Pf Consider

$$\|\Sigma_{k+1} - \Sigma_k\|_\infty$$

$$= \max_{t \in I} |\Sigma_{k+1}(t) - \Sigma_k(t)|$$

$$\leq \max_{t \in I} \left| \int_{t_0}^t |F(\Sigma_k(s)) - F(\Sigma_{k-1}(s))| ds \right|$$

$$\leq \max_{t \in I} |t - t_0| \cdot \max_{s \in I} |F(\Sigma_k(s)) - F(\Sigma_{k-1}(s))|$$

Noting that

$$\begin{aligned} |F(\underline{x}_k(s)) - F(\underline{x}_{k-1}(s))| &\leq L |\underline{x}_k(s) - \underline{x}_{k-1}(s)| \\ &\leq L \|\underline{x}_k - \underline{x}_{k-1}\|_\infty \quad \forall s \in I, \end{aligned}$$

we get

$$\begin{aligned} \|\underline{x}_{k+1} - \underline{x}_k\|_\infty &\leq a L \|\underline{x}_k - \underline{x}_{k-1}\|_\infty \\ &\leq (aL)^k \underbrace{\|\underline{x}_1 - \underline{x}_0\|_\infty}_\delta \\ &\leq 2^{-k} \delta. \end{aligned}$$

So for $m > n$,

$$\begin{aligned} \|\underline{x}_m - \underline{x}_n\|_\infty &= \|\underline{x}_m - \underline{x}_{m-1} + \underline{x}_{m-1} - \underline{x}_{m-2} + \underline{x}_{m-2} - \underline{x}_{m-3} + \dots \\ &\quad - \underline{x}_{n+1} + \underline{x}_{n+1} - \underline{x}_n\|_\infty \\ &\leq \sum_{k=n}^{m-1} \|\underline{x}_{k+1} - \underline{x}_k\|_\infty \\ &\leq \delta \cdot \sum_{k=n}^{m-1} 2^{-k} \\ &\rightarrow 0 \quad \text{as } m, n \rightarrow \infty. \end{aligned}$$

Hence $\lim_{n \rightarrow \infty} \Sigma_n =: \Sigma \in C(I, \mathbb{R}^n)$,

so that $\forall t \in I$,

$$\Sigma(t) = \lim_{n \rightarrow \infty} \Sigma_n(t)$$

$$= \Sigma_0 + \lim_{n \rightarrow \infty} \int_{t_0}^t F(\Sigma_n(s)) ds$$

$$= \Sigma_0 + \int_{t_0}^t F(\Sigma(s)) ds.$$

Last equality follows from

$$\Sigma_n(I) \subset \overline{B}_\delta(\Sigma_0) \quad \forall n \Rightarrow \Sigma(I) \subset \overline{B}_\delta(\Sigma_0)$$

$$(\|\Sigma - \Sigma_0\|_\infty = \lim_{n \rightarrow \infty} \|\Sigma_n - \Sigma_0\|_\infty \leq \delta),$$

$$\text{and } \left| \int_{t_0}^t F(\Sigma_n(s)) ds - \int_{t_0}^t F(\Sigma(s)) ds \right|$$

$$\leq \underbrace{\left| \int_{t_0}^t (F(\Sigma_n(s)) - F(\Sigma(s))) ds \right|}_{\leq L \|\Sigma_n - \Sigma\|_\infty \rightarrow 0}$$

$$\rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

lastly, $\mathcal{X}'(t) = \frac{d}{dt} \left(\mathcal{X}_0 + \int_0^t F(\mathcal{X}(s)) ds \right) = F(\mathcal{X}(t))$
and $F(\mathcal{X}(\cdot)) \in C(I, \mathbb{R}^n)$, so $\mathcal{X} \in C^1(I, \mathbb{R}^n)$.

Uniqueness: Suppose $\mathcal{Y} \in C^1(I, \mathbb{R}^n)$ is
another solution on I , meaning
 $\mathcal{Y}(t) = \mathcal{X}_0 + \int_{t_0}^t F(\mathcal{Y}(s)) ds$.

Claim It must also hold that $\mathcal{Y}(t) \in \overline{B}_\delta(\mathcal{X}_0)$
for all $t \in I$. Pf. Suppose not. Let
then $\bar{a} \in (-a, a)$ be the smallest value
in magnitude so that

$$|\mathcal{Y}(t_0 + \bar{a}) - \mathcal{X}_0| = \delta \text{ and}$$

$$\mathcal{Y}(t) \in \overline{B}_\delta(\mathcal{X}_0) \quad \forall t \in [t_0 - \bar{a}, t_0 + \bar{a}].$$

Then $|F(\mathcal{Y}(s))| \leq M \quad \forall s \in [t_0 - \bar{a}, t_0 + \bar{a}]$,
and we get contradiction

$$\delta = |\mathcal{Y}(t_0 + \bar{a}) - \mathcal{X}_0| \leq \left| \int_{t_0}^{t_0 + \bar{a}} |F(\mathcal{Y}(s))| ds \right|$$

$$\leq |\bar{a}| M < a M \leq \delta$$

(End proof of claim)

For any $t \in I$, we have

$$\begin{aligned} |X(t) - Y(t)| &= \left| \int_{t_0}^t (F(X(s)) - F(Y(s))) ds \right| \\ &\leq L \left| \int_{t_0}^t |X(s) - Y(s)| ds \right| \\ &\leq aL \|X - Y\|_{\infty} \\ &\leq \frac{1}{2} \|X - Y\|_{\infty} \end{aligned}$$

Taking $\max_{t \in I}$ yields

$$\|X - Y\|_{\infty} \leq \frac{1}{2} \|X - Y\|_{\infty} \Rightarrow X = Y.$$

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Note The map $K: \underbrace{C(I, \overline{B}_\delta(X_0))}_{=: V} \rightarrow C(I, \overline{B}_\delta(X_0))$
given by

$$K[X](\cdot) := X_0 + \int_{t_0}^{\cdot} F(X(s)) ds$$

is a contraction on V :

$$\|K[X] - K[Y]\|_\infty \leq \frac{1}{2} \|X - Y\|_\infty \quad \forall X, Y \in V$$

& contraction principle $\Rightarrow \exists ! X \in V \subset C(I, \mathbb{R}^n)$
s.t. the sequence

$$X_1 = K(X_0), X_2 = K(X_1), \dots$$

converges to X in $(C(I, \mathbb{R}^n), \|\cdot\|_\infty)$.

Ex 1 Let $A \in \mathbb{R}^{n \times n}$ and consider

$$\underline{X}' = A \underline{X} = F(\underline{X})$$

$$\underline{X}(0) = \underline{X}_0$$

F is globally Lipschitz:

$$\begin{aligned} |F(\underline{X}) - F(\underline{Y})| &= |A(\underline{X} - \underline{Y})| \\ &\leq \|A\|_2 |\underline{X} - \underline{Y}| \end{aligned}$$

Thm 1 $\Rightarrow$ For some $I = [-a, a]$,
the ODE has a unique sol
 $\underline{X} \in C^1(I, \mathbb{R}^d)$

Q: Know already that $\underline{X}(t) = e^{At} \underline{X}_0$
is unique sol on $C^1(\mathbb{R}, \mathbb{R}^d)$. When
can we extend Thm 1 from local
to globally unique solutions?