

Recall

Theorem 1 Let $D \subset \mathbb{R}^n$ be a domain and $F \in C^1(D, \mathbb{R}^n)$. Then the IVP

$$\mathbf{x}' = F(\mathbf{x})$$

$$\mathbf{x}(t_0) = \mathbf{x}_0 \in D$$

has a unique local solution

$\mathbf{x} \in C^1([t_0 - a, t_0 + a], \mathbb{R}^n)$ for some $a > 0$.

Q: When can we extend to global uniqueness?

Plan

- Extension of solutions
- Existence & uniqueness for nonautonomous IVP.

(Example: global solutions do not always exist.)

Ex 1 $x' = x^2$
 $x(0) = 1$

$f \in C^\infty(\mathbb{R}) \Rightarrow \exists$ a locally unique
solution near $t=0$, namely
 $x(t) = \frac{1}{1-t}$ (use separation of var)

Solution is unique on $(-\infty, T_+) \subset \mathbb{R}$
where $T_+ = 1$.

Extension of local solutions

Theorem 2 If $F \in C^1(D, \mathbb{R}^n)$,

then the IVP

$$X' = F(X)$$

$$X(t_0) = X_0 \in D$$

has a unique C^1 -solution defined on a maximal interval $I(X_0, t_0) = (T_-(X_0, t_0), T_+(X_0, t_0))$

where $T_{\pm}(X_0, t_0) \in [-\infty, \infty]$. No solution can be extended any further in any direction, and if $T_+ < \infty$, then $X(t)$ approaches ∂D as $t \uparrow T_+$ (and similarly for $t \downarrow T_-$).

Proof idea

Thm 1 $\Rightarrow$ Sol is unique on

$I_0 = [t_0 - a_0, t_0 + a_0]$. Repeat with $I \subset$

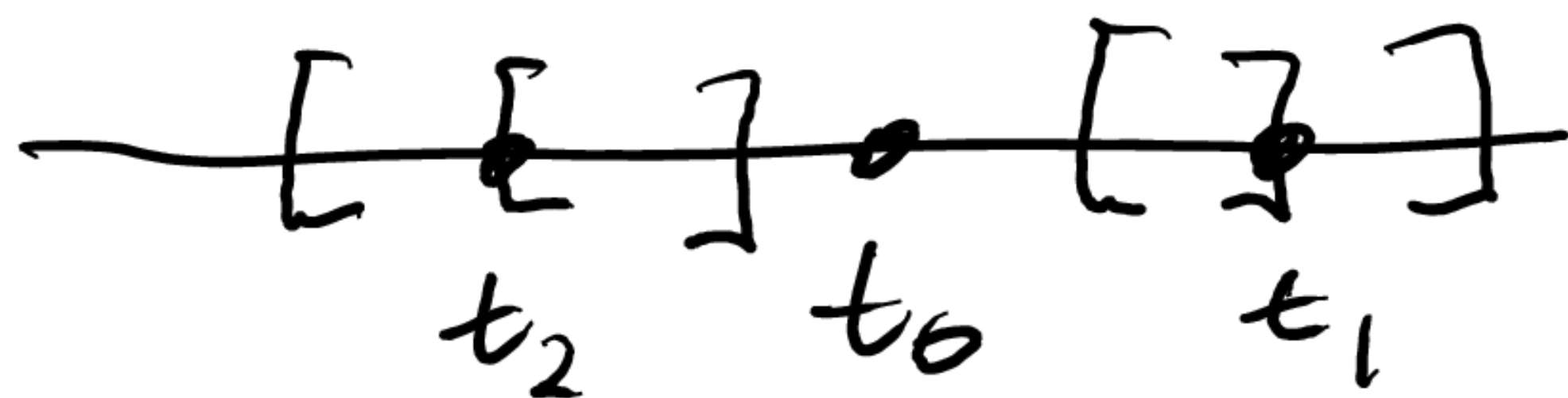
$X(t_0 \pm a_0)$ at $t_1 = t_0 + a_0$, $t_2 = t_0 - a_0$

to extend uniqueness to

$$I_1 = [t_1 - a_1, t_1 + a_1]$$

$$I_2 = [t_2 - a_2, t_2 + a_2]$$

for some $a_0, a_1, a_2 > 0$



and solutions agree on overlaps
 $I_0 \cap I_1$ & $I_0 \cap I_2$
 so must be one solution.

yields unique solution on $I = \bigcup_{i=1}^3 I_i$,

Iterating argument as long as possible,

$$I = \bigcup_{i=1}^{\infty} I_i = (T_-, T_+) \subset \mathbb{R}$$

and this is an open set, as if it were closed, say $[T_-, T_+]$, then necessarily

$X(T_+) \in D$ (as $X(T_+)$ then still is a solution), which by Thm $\Rightarrow$

$\exists a_+ > 0$ and unique sol on

$[T_+ - a_+, T_+ + a_+]$, contradicting maximality. $\blacksquare$

Corollary 1 If $F \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ is globally Lipschitz, then

$$\mathbf{X}' = F(\mathbf{X})$$

$$\mathbf{X}(t_0) = \mathbf{X}_0 \in \mathbb{R}^n$$

has a globally unique solution $\mathbf{X} \in C^1(\mathbb{R}, \mathbb{R}^d)$.

Proof idea The domain $D = \mathbb{R}^n$. Following the proof of Thm 1, we can now set $a = \frac{1}{2L}$. The additional constraint $a \leq \frac{\delta}{M}$ was used to ensure that $\mathbf{X}_{k+1}(t) = (\mathbf{X}_0 + \int_{t_0}^t F(\mathbf{X}_k(s)) ds) \in D$ for all $t \in [t_0 - a, t_0 + a]$. It is not needed when $D = \mathbb{R}^n$.

We obtain a unique solution $\mathbf{X} \in C^1(I_0, \mathbb{R}^d)$ where $I_0 = [t_0 - a, t_0 + a]$. By iteration we get a unique solution also on $I_1 = [t_1 - a, t_1 + a]$ where $t_1 = t_0 + a$ and on $I_2 = [t_2 - a, t_2 + a]$ $t_2 = t_0 - a$

And solutions agree on overlaps, so extension
on $I = \bigcup_{i=0}^{\infty} I_i = \mathbb{R}$ becomes one unique

solution. $\mathbb{R}$

(key in above argument: same $a > 0$ for every
interval I_i .)

Ex 2 Let $A \in \mathbb{R}^{n \times n}$ and consider

$$\underline{x}' = A \underline{x} = F(\underline{x})$$

$$\underline{x}(0) = \underline{x}_0$$

F is globally Lipschitz:

$$|F(\underline{x}) - F(\underline{y})| = |A(\underline{x} - \underline{y})| \leq \underbrace{\|A\|_0}_{=L} |\underline{x} - \underline{y}|$$

Corollary 1 $\Rightarrow \exists! \underline{x} \in C^1(\mathbb{R}; \mathbb{R}^d)$ solving
the ODE (which we know is $\underline{x} = e^{At} \underline{x}_0$).

Ex 3 Showed in Lecture 6

exercise 1.11g that $x' = f(x) = \begin{cases} (1+x)^2 & x \geq 0 \\ (1-x)^2 & x < 0 \end{cases}$

has solution $x(t) = \frac{t}{1-|t|}$ $t \in (-1, 1)$

with $\lim_{|t| \uparrow} |x(t)| = \infty$.

As $f \in C^1(\mathbb{R})$, Corollary 1 $\Rightarrow$

x is unique C^1 solution

on $(T_-, T_+) = (-1, 1)$.

Next

Existence & uniqueness for non-autonomous
ODE.

Def For an open interval $I \subset \mathbb{R}$ and a domain $D \subset \mathbb{R}^n$, let

$$C^{0,1}(I \times D, \mathbb{R}^n) := \{F: I \times D \rightarrow \mathbb{R}^n \mid$$

$$f_i, \partial_{x_j} f_i \in C(I \times D) \quad \forall i, j \in \{1, \dots, n\}\}$$

($F = (f_1(t, \underline{x}), \dots, f_n(t, \underline{x}))$ is continuous in t and continuously differentiable in $x_1, \dots, x_n$)

Def $F \in C(I \times D, \mathbb{R}^n)$ is locally

Lipschitz in $\underline{x}$ if for every $(t, \underline{x}) \in I \times D$

there exists compact set

$$V_\delta = [t - \delta, t + \delta] \times \overline{B}_\delta(\underline{x}) \subset I \times D$$

s.t. $F|_{V_\delta}$ is Lipschitz in $\underline{x}$, meaning

there exists an $L > 0$ s.t.

$$|F(s, \underline{Y}) - F(s, \underline{Z})| \leq L |\underline{Y} - \underline{Z}| \quad (1)$$

for all pairs $(s, \underline{Y}), (s, \underline{Z}) \in V_\delta$.

If (1) holds for some $L > 0$ on all

of $I \times D$, then F is globally Lipschitz in $\mathbb{R}$ (on $I \times D$).

Lemma If $I \subset \mathbb{R}$ is open, then $F \in C^{0,1}(I \times D, \mathbb{R}^n) \Rightarrow F$ is locally Lipschitz in $\mathbb{R}$.

Proof

$$\text{Let } D_x F(s, \mathbb{I}) = \begin{bmatrix} \partial_{x_1} f_1 & \dots & \partial_{x_n} f_1 \\ \vdots & & \vdots \\ \partial_{x_1} f_n & & \partial_{x_n} f_n \end{bmatrix} (s, \mathbb{I})$$

and for $(s, \mathbb{I}), (s, \mathbb{Z}) \in V_\delta$, let

$$\mathbb{X}(r) = \mathbb{I} + r(\mathbb{Z} - \mathbb{I}) \quad r \in [0, 1].$$

Then

$$|F(s, \mathbb{Z}) - F(s, \mathbb{I})| = \left| \int_0^1 \frac{d}{dr} F(s, \mathbb{X}(r)) dr \right|$$

$$\leq \int_0^1 \|D_x F(s, \mathbb{X}(r)) (\mathbb{Z} - \mathbb{I})\| dr$$

$$\leq \int_0^1 \|D_x F(s, \mathbb{X}(r))\|_2 dr |\mathbb{Z} - \mathbb{I}|$$

$$\leq L |\mathbb{Z} - \mathbb{I}|, \text{ where } L = \max_{(s, \mathbb{I}) \in V_\delta} \|D_x F(s, \mathbb{I})\|_2$$

Theorem 3 Consider IVP

$$\left. \begin{aligned} \mathbf{x}'(t) &= F(t, \mathbf{x}) \\ \mathbf{x}(t_0) &= \mathbf{x}_0 \in D \subset \mathbb{R}^n \end{aligned} \right\} \quad (2)$$

where $F: I \times D \rightarrow \mathbb{R}^n$, $I \subset \mathbb{R}$ open.

If $F \in C^{0,1}(I \times D, \mathbb{R}^n)$, then there exists an $\alpha > 0$ and a locally unique solution $\mathbf{x} \in C^1([t_0 - \alpha, t_0 + \alpha], \mathbb{R}^n)$.

Proof idea Given $(t_0, \mathbf{x}_0)$, find a

$\delta > 0$, s.t. for $V_\delta = [t_0 - \delta, t_0 + \delta] \times \overline{B}_\delta(\mathbf{x}_0)$,

$f|_{V_\delta}$ is Lipschitz in $\mathbf{x}$ with constant

$L > 0$.

Let $M := \max_{(s, \mathbf{z}) \in V_\delta} |f(s, \mathbf{z})|$ and set

$$\alpha = \min\left(\frac{\delta}{M}, \frac{1}{2L}\right).$$

For $|t - t_0| \leq \alpha$,

$$\underline{X}_{k+1}(t) = \underline{X}_0 + \int_{t_0}^t F(s, \underline{X}_k(s)) ds \quad k=0,1,\dots$$

satisfies $(\underline{X}_k) \subset C([t_0 - \alpha, t_0 + \alpha], \overline{B}_\delta(\underline{X}_0))$,

and for $k \geq 2$,

$$|\underline{X}_{k+1}(t) - \underline{X}_k(t)| \leq \left| \int_{t_0}^t L |\underline{X}_k(s) - \underline{X}_{k-1}(s)| ds \right|$$

$$\leq \underbrace{|t - t_0|}_{\leq \alpha} L \|\underline{X}_k - \underline{X}_{k-1}\|_\infty$$

$$\leq \frac{1}{2} \|\underline{X}_k - \underline{X}_{k-1}\|_\infty$$

where $\|\underline{Y}\|_\infty := \max_{t \in [t_0 - \alpha, t_0 + \alpha]} |\underline{Y}(t)|$.

$$\Rightarrow \|\underline{X}_{k+1} - \underline{X}_k\|_\infty \leq 2^{-k+1} \|\underline{X}_1 - \underline{X}_0\|_\infty,$$

which can be used to show that

$(\underline{X}_k)$ is Cauchy in $C([t_0 - \alpha, t_0 + \alpha], \mathbb{R}^n)$

and $\underline{X} = \lim_{k \rightarrow \infty} \underline{X}_k$ solves IVP on

$$[t_0 - \alpha, t_0 + \alpha].$$

Local uniqueness proved as in autonomous case (by contraction argument).

Global solutions

- The nonautonomous IVP (2) has a unique C^1 -solution defined on a maximal interval $I_{(x_0, t_0)} = (T_-, T_+) \subset I$.

If e.g. $T_+ < \max_I t$, then

$x(t)$ approaches ∂D as $t \uparrow T_+$.

- If $F \in C^{0,1}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ in (2) is globally Lipschitz in x , then the IVP has a globally unique solution $x \in C^1(\mathbb{R}, \mathbb{R}^n)$.

Ex 4 $x' = tx^2 =: f(t, x)$, $f \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$
 $x(0) = 1/2$

separation of variables

$$\int \frac{1}{x^2} dx = \int t dt$$

$$\frac{1}{x} = 2 - t^2/2$$

yields unique solution $x(t) = \frac{1}{2 - t^2/2}$

on $I_{(x_0=1/2, t_0=0)} = (-2, 2)$

For general $x_0 > 0$,

$$I_{(x_0, t_0=0)} = (-\sqrt{2} x_0^{-1/2}, \sqrt{2} x_0^{-1/2})$$

and for $x_0 \leq 0$, $I_{(x_0, t_0)} = \mathbb{R}$.

(Treat $x_0 = 0$ and $x_0 < 0$ separately.)

Ex 5 $\underline{X}'(t) = A(t)\underline{X}(t) =: F(t, \underline{X})$

$$\underline{X}(0) = \underline{X}_0 \in \mathbb{R}^n$$

and $A \in C(\mathbb{R}, \mathbb{R}^{n \times n})$ with

$$\|A(t)\|_2 \leq C \quad \forall t \in \mathbb{R}.$$

Then $F \in C^{0,1}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is globally Lipschitz in $\underline{X}$
on $\mathbb{R} \times \mathbb{R}^n \Rightarrow \exists!$ solution $\underline{X} \in C^1(\mathbb{R}, \mathbb{R}^n)$.