

Recall that

$$\left. \begin{aligned} \mathbf{X}'(t) &= F(t, \mathbf{X}) \\ \mathbf{X}(t_0) &= \mathbf{X}_0 \in D \end{aligned} \right\} (1)$$

with $F \in C^{0,1}(I \times D, \mathbb{R}^n)$ has a unique solution $\mathbf{X} \in C^1((T_-, T_+), \mathbb{R}^n)$.

If $F \in C^{0,1}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is globally

Lipschitz, meaning

$$L = \sup_{(s, \mathbf{X}) \neq (s, \mathbf{Y}) \in \mathbb{R} \times \mathbb{R}^n} \frac{|F(s, \mathbf{X}) - F(s, \mathbf{Y})|}{|\mathbf{X} - \mathbf{Y}|} < \infty,$$

then unique solution is global $T_{\pm} = \pm\infty$.

Q: How sensitive is (1) wrt perturbations in $\mathbf{X}_0$: $|\mathbf{X}(t; \mathbf{X}_0) - \mathbf{X}(t; \tilde{\mathbf{X}}_0)| = ?$

Is it even true that $\lim_{\tilde{\mathbf{X}}_0 \rightarrow \mathbf{X}_0} \mathbf{X}(t; \tilde{\mathbf{X}}_0) = \mathbf{X}(t; \mathbf{X}_0)$?

Without loss of generality, we consider IVP (1) with $t_0 = 0$ (Solve IVP for $\gamma(t) := \mathbb{X}(t_0 + t)$).

$$\text{Let } \phi_t(\mathbb{X}_0) := \mathbb{X}(t; \mathbb{X}_0)$$

denote the flow for $t \in (T_-(\mathbb{X}_0), T_+(\mathbb{X}_0))$.

$$\text{Then } \phi_t(\mathbb{X}_0) = \underbrace{\mathbb{X}_0}_{\phi_0(\mathbb{X}_0)} + \int_0^t F(s, \phi_s(\mathbb{X}_0)) ds \quad (2)$$

Theorem 1 Consider (1) with $t_0 = 0$.

Let $\mathbb{X}_0, \mathbb{Y}_0 \in D$ and $F \in C^{0,1}(I \times D, \mathbb{R}^n)$ be globally Lipschitz in $\mathbb{X}$ on $I \times D$ with constant $L > 0$. Then

$$|\phi_t(\mathbb{X}_0) - \phi_t(\mathbb{Y}_0)| \leq \underbrace{|\mathbb{X}_0 - \mathbb{Y}_0|}_{\text{both}} e^{L|t|} \quad (3)$$

for all times when $\checkmark$ solutions $\phi_t(\mathbb{Y}_0), \phi_t(\mathbb{X}_0) \in D$ exist, i.e. $t \in (T_-(\mathbb{X}_0), T_+(\mathbb{X}_0)) \cap (T_-(\mathbb{Y}_0), T_+(\mathbb{Y}_0))$.

Proof

$$|\phi_t(\mathbb{X}_0) - \phi_t(\mathbb{Y}_0)| \stackrel{(2)}{\leq} |\mathbb{X}_0 - \mathbb{Y}_0| + \int_0^t |F(s, \phi_s(\mathbb{X}_0)) - F(s, \phi_s(\mathbb{Y}_0))| ds$$

$$\leq |\phi_0(x) - \phi_0(y_0)| + \int_0^t L |\phi_s(x_0) - \phi_s(y_0)| ds \quad (4)$$

Grönwall's inequality on (4) $\Rightarrow$ (3).

Proof for $t < 0$. Consider $\psi_t(x_0) := \phi_{-t}(x_0)$.



Grönwall's inequality

If $f \in C([0, T])$ is non-negative and satisfies

$$f(t) \leq C + \int_0^t L f(s) \quad \forall t \in [0, a] \quad (5)$$

for some $C, L \geq 0$ and $a \in (0, T]$,

then $f(t) \leq C e^{Lt} \quad \forall t \in [0, a]$.

Remark Apply to (4) with
 $f(s) = |\phi_s(x_0) - \phi_s(y_0)|$ in above proof.

Proof Let $I(t) = \int_0^t f(s) ds$.

Then $I'(t) = f(t)$, so (5) becomes

$$I'(t) \leq C + LI(t) \quad \left| e^{-Lt} \text{ integrating factor} \right.$$

$$\Rightarrow \underbrace{\frac{d}{dt} (e^{-Lt} I)}_{g(t)} \leq \underbrace{C e^{-Lt}}_{h(t)} \quad \left| \int_0^t \right.$$

(and use that if $g \leq h$, then $\int h - g dt \geq 0$)

$$\Rightarrow e^{-Lt} I(t) \leq \underbrace{I(0)}_{=0} + \frac{C}{L} (1 - e^{-Lt})$$

$$\Rightarrow I(t) \leq \frac{C}{L} (e^{Lt} - 1)$$

Returning to (5)

$$f(t) \leq C + LI(t) \leq C e^{Lt}$$

If instead Σ and $\mathcal{I}$ solves (1) with $\Sigma(t_0) = \Sigma_0$, $\mathcal{I}(t_0) = \mathcal{I}_0$ for a general $t_0 \in I$, then (3) becomes

$$|\Sigma(t) - \mathcal{I}(t)| \leq |\Sigma_0 - \mathcal{I}_0| e^{L|t-t_0|}$$

Ex 1

$$x' = 3x = f(x)$$

$$L = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} = 3.$$

$$\phi_t(x_0) = x_0 e^{3t}$$

$$|\phi_t(x_0) - \phi_t(y_0)| \leq |x_0 - y_0| e^{3|t|}$$

Ex 2

$$\Sigma' = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_A \Sigma$$

$$L = \sup_{\underline{X} \neq \underline{Z} \in \mathbb{R}^2} \frac{|A\underline{X} - A\underline{Z}|}{|\underline{X} - \underline{Z}|} = \|A\|_2 = 1$$

Write $\underline{X}_0 = (x_1^0, x_2^0)$. Then

$$\phi_t(\underline{X}_0) = x_1^0 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + x_2^0 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

And for all $t \in \mathbb{R}$,

$$|\phi_t(\underline{X}_0) - \phi_t(\underline{Z}_0)| = |\underline{X}_0 - \underline{Z}_0|$$

$$\leq |\underline{X}_0 - \underline{Z}_0| e^{|t|}.$$

Theorem 1 shows that if F is globally Lipschitz in $\underline{X}$, then $\phi_t(\underline{X}_0) = \phi(t, \underline{X}_0)$ is C^1 in time and locally Lipschitz in $\underline{X}$. But ϕ is in fact also C^1 on $\underline{X}_0$ (see §17.6).

Example 3 shows estimate (3) is not always sharp. Better first-order estimate:

$$\phi(t, \bar{x}_0 + \Delta \bar{x}_0) = \phi(t, \bar{x}_0) + D_{\bar{x}} \phi(t, \bar{x}_0) \Delta \bar{x}_0 + O(|\Delta \bar{x}_0|^2) \quad (6)$$

where $\phi(t, \bar{x}_0) := \phi_t(\bar{x}_0)$ and

$$D_{\bar{x}} \phi(t, \bar{x}_0) = \begin{bmatrix} \phi'_{x_1} & \dots & \phi'_{x_n} \\ \vdots & & \vdots \\ \phi''_{x_1} & \dots & \phi''_{x_n} \end{bmatrix} \in \mathbb{R}^{n \times n}$$

with $\bar{x}_0 = (x_1^0, \dots, x_n^0)$.

IVP for $U(t) := D_{\bar{x}} \phi(t, \bar{x}_0)$

$$\phi(t, \bar{x}_0) = \bar{x}_0 + \int_0^t F(s, \phi(s, \bar{x}_0)) ds$$

$$\Rightarrow \underbrace{D_{\bar{x}} \phi(t, \bar{x}_0)}_{U(t)} = I + \int_0^t D_{\bar{x}} F(s, \phi(s, \bar{x}_0)) \underbrace{D_{\bar{x}} \phi(s, \bar{x}_0)}_{U(s)} ds$$

Variational equation:

$$U'(t) = B(t, x_0) U(t)$$

$$U(0) = I$$

where $B(t, x_0) = D_x F(t, \phi(t, x_0))$
is independent of U and ODE is linear.

Ex 3

$$x' = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_A x = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} =: F(x)$$

$$D_x F(x) = A \quad \forall x \in \mathbb{R}^2$$

$$\Rightarrow B(t, x_0) = A.$$

$$U' = A U \quad \text{yields}$$

$$U(0) = I$$

$$\begin{bmatrix} u_{11}' & u_{12}' \\ u_{21}' & u_{22}' \end{bmatrix} = A \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \quad U(0) = I.$$

solve column-wise:

$$\Rightarrow U(t) = e^{At} U(0) = e^{At} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

Variational (first-order) approximation:

$$\begin{aligned} \phi(t, \bar{x}_0 + \Delta \bar{x}_0) &= \phi(t, \bar{x}_0) + \underbrace{D_{\bar{x}_0} \phi(t, \bar{x}_0)}_{U(t)} \Delta \bar{x}_0 + O(\Delta \bar{x}^2) \\ &\approx \phi(t, \bar{x}_0) + e^{At} \Delta \bar{x}_0 \end{aligned}$$

But we know solution of ODE from earlier:

$$\phi(t, \bar{x}_0) = e^{At} \bar{x}_0,$$

so for this linear problem the first-order approximation is exact, as

$$\underbrace{\phi(t, \bar{x}_0 + \Delta \bar{x}_0)}_{e^{At} (\bar{x}_0 + \Delta \bar{x}_0)} = \underbrace{e^{At} \bar{x}_0}_{\phi(t, \bar{x}_0)} + \underbrace{e^{At} \Delta \bar{x}_0}_{D_{\bar{x}_0} \phi(t, \bar{x}_0)}$$

Variational approximations at EP $\underline{x}^*$

Consider autonomous ODE

$$\underline{x}' = F(\underline{x})$$

with an EP 0 (meaning $\phi(t, 0) = 0$).

To approximate $\phi(t, \underline{x}_0)$ for $\underline{x}_0$ near 0 ,
note that

$$\phi(t, \underline{x}_0) = \underbrace{\phi(t, 0)}_{=0} + D_{\underline{x}} \phi(t, 0) \underline{x}_0 + O(|\underline{x}_0|^2),$$

where $U(t) = D_{\underline{x}} \phi(t, 0)$ solves ODE

$$U' = D_{\underline{x}} F(\underbrace{\phi(t, 0)}_{=0}) U = D_{\underline{x}} F(0) U$$

$$U(0) = I$$

solution $U(t) = e^{DF(0)t}$, so

$$\phi(t, \underline{x}_0) = e^{DF(0)t} \underline{x}_0 + O(|\underline{x}_0|^2)$$

$\tilde{X}(t) := e^{DF(0)t} X_0$ is called the linearized system near 0, and it satisfies the ODE

$$\tilde{X}'(t) = DF(0) \tilde{X}(t)$$

$$\tilde{X}(0) = X_0,$$

and $|\phi(t, X_0) - \tilde{X}(t)| = O(|X_0|^2).$

For $X' = F(X)$ with an EP X^* and X_0 near X^* , the above generalizes to

$$\tilde{X}' = DF(X^*)(\tilde{X}(t) - X^*)$$

$$\tilde{X}(0) = X_0$$

and $|\phi(t, X_0) - \tilde{X}(t)| = O(|X_0 - X^*|^2)$

Theorem (Linearization thm)

If $X' = F(X)$ has an EP X^* and

$DF(X^*)$ is hyperbolic ($\operatorname{Re}(\lambda) \neq 0 \forall EV$),
then $\tilde{X}' = DF(X^*)(\tilde{X} - X^*)$ is conjugate to
 X in a neighborhood of X^* .

Nonlinear example

$$\mathbb{X}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + y^2 \\ -y \end{bmatrix} =: F(\mathbb{X}) \quad (7)$$

Q: How is $\phi(t, \mathbb{X}_0)$ for $\mathbb{X}_0$ near EP 0?

Linearized system at 0:

$$DF(\mathbb{X}) = \begin{bmatrix} 1 & 2y \\ 0 & -1 \end{bmatrix}, \quad DF(0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\tilde{\mathbb{X}}'(t) = DF(0)\tilde{\mathbb{X}}$$

$$\tilde{\mathbb{X}}(0) = \mathbb{X}_0$$

$$\Rightarrow \tilde{\mathbb{X}}(t) = e^{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}t} \mathbb{X}_0 = \begin{bmatrix} x_0 e^t \\ y_0 e^{-t} \end{bmatrix}$$

Compare to exact solution of (7):

$$y' = -y \Rightarrow y(t) = y_0 e^{-t}$$

$$x' = x + y^2 = x + y_0^2 e^{-2t} \quad | \cdot e^{-t}$$

$$\Rightarrow (x e^{-t})' = y_0^2 e^{-3t} \quad | \int_0^t$$

$$x(t) = \left(x_0 + \frac{y_0^2}{3}\right) e^t - \frac{y_0^2}{3} e^{-2t}$$

$$\underline{x}(t) = \begin{bmatrix} \left(x_0 + \frac{y_0^2}{3}\right) e^t - \frac{y_0^2}{3} e^{-2t} \\ y_0 e^{-t} \end{bmatrix}$$

$$\begin{aligned} \Rightarrow |\underline{x}(t) - \tilde{\underline{x}}(t)| &= \left| \begin{bmatrix} \frac{y_0^2}{3} (e^t - e^{-2t}) \\ 0 \end{bmatrix} \right| \\ &= \mathcal{O}(|x_0|^2) \end{aligned}$$

for bounded t -values.

NB!

Asymptotic behavior can differ outside neighborhood:

For $\underline{x}_0 = \begin{bmatrix} 0 \\ \delta \end{bmatrix}$ with $\delta \neq 0$, we have

$$\tilde{\underline{x}}(t) = \begin{bmatrix} 0 \\ \delta e^{-t} \end{bmatrix} \text{ and } \underline{x}(t) = \begin{bmatrix} \frac{\delta^2}{3} (e^t - e^{-2t}) \\ \delta e^{-t} \end{bmatrix}$$

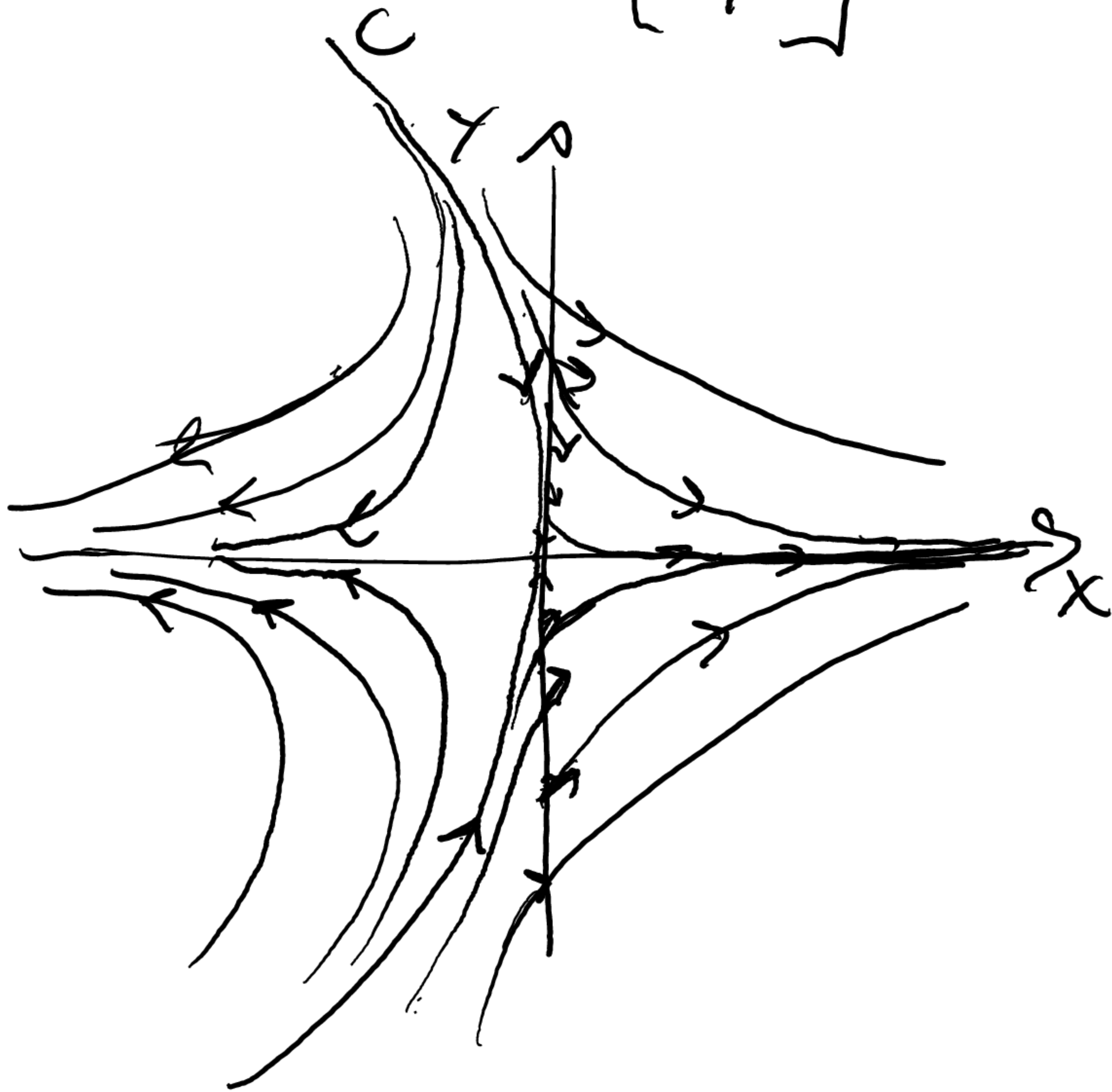
where

$$|\tilde{\underline{x}}(t)| \rightarrow 0 \text{ and } |\underline{x}(t)| \rightarrow \infty \text{ as } t \rightarrow \infty.$$

y -axis is a stable direction/curve only for the linear problem.

Nevertheless, 0 is a saddle point both for $\tilde{\underline{x}}$ and for $\underline{x}$ dynamics (as $\underline{x}$ also has a stable curve).

Phase plane for $F(\underline{x}) = \begin{bmatrix} x + y^2 \\ -y \end{bmatrix}$



Stable curve for nonlinear problem:
consider curve $C = \{(x, y) \mid x + y^2/3 = 0\}$

For $(x_0, y_0) \in C$, we get

$$\underline{X}(t) = \begin{bmatrix} \frac{-y_0^2}{3} e^{-2t} + \underbrace{\left(x_0 + \frac{1}{3}y_0^2\right)}_{=0} e^t \\ y_0 e^{-t} \end{bmatrix}$$

satisfying

$$1. x(t) + \frac{1}{3}(y(t))^2 = 0 \quad \forall t$$

and

$$2. |\bar{x}(t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

When $\bar{x}$ satisfies 1 & 2, we say it has a stable solution curve C .

Global conjugate dynamics (for this problem)

Consider homeomorphism

$$H(\bar{x}) = \begin{bmatrix} x + \frac{1}{3}y^2 \\ y \end{bmatrix} =: \begin{bmatrix} u \\ v \end{bmatrix} = U$$

Then the dynamics

$$u' = x' + \frac{1}{3} 2y y' = x + \frac{y^2}{3} = u$$

$$v' = y' = -y = -v$$

i.e. $U' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} U$ (linear dynamics like $\bar{x}$)

is conjugate to $\bar{x}(t, \bar{x}_0)$ via H :

$$H(\bar{x}(t; \bar{x}_0)) = U(t; H(\bar{x}_0)).$$

NB! Linearization does not always preserve even type of EP when $DF(\bar{X}^*)$ is not hyperbolic.

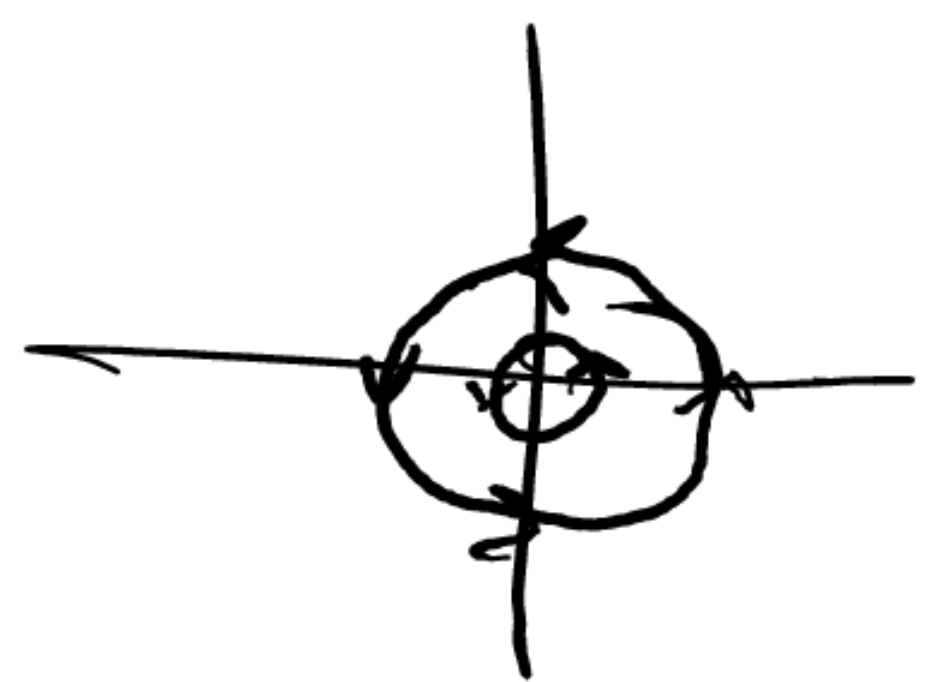
Ex For $\varepsilon \neq 0$ consider

$$\bar{X} = \begin{bmatrix} -y + \varepsilon x(x^2 + y^2) \\ x + \varepsilon y(x^2 + y^2) \end{bmatrix} =: F(\bar{X})$$

EP: $\bar{X}^* = 0$. Linearized problem

$$DF(\bar{X}) = \begin{bmatrix} \varepsilon(3x^2 + y^2) & 2\varepsilon xy - 1 \\ 1 + 2\varepsilon xy & \varepsilon(3y^2 + x^2) \end{bmatrix}$$

$$\tilde{\bar{X}}' = DF(0)\tilde{\bar{X}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tilde{\bar{X}}.$$



EV $\lambda_{\pm} = \pm i \Rightarrow 0$ is a circular center for $\tilde{\bar{X}}$.
(stable EP)

Exact solution in polar coordinates:

$$x(t) = r(t) \cos \theta(t)$$

$$y(t) = r(t) \sin \theta(t)$$

$$(r, \theta) \in (0, \infty) \times \underbrace{[-\pi, \pi]}_{\mathbb{T}}$$

ODE:

$$x' = -y + \varepsilon x(x^2 + y^2) = -r \sin \theta + \varepsilon r^3 \cos \theta$$

$$y' = x + \varepsilon y(x^2 + y^2) = r \cos \theta + \varepsilon r^3 \sin \theta$$

and $x' = r' \cos \theta - r \theta' \sin \theta$

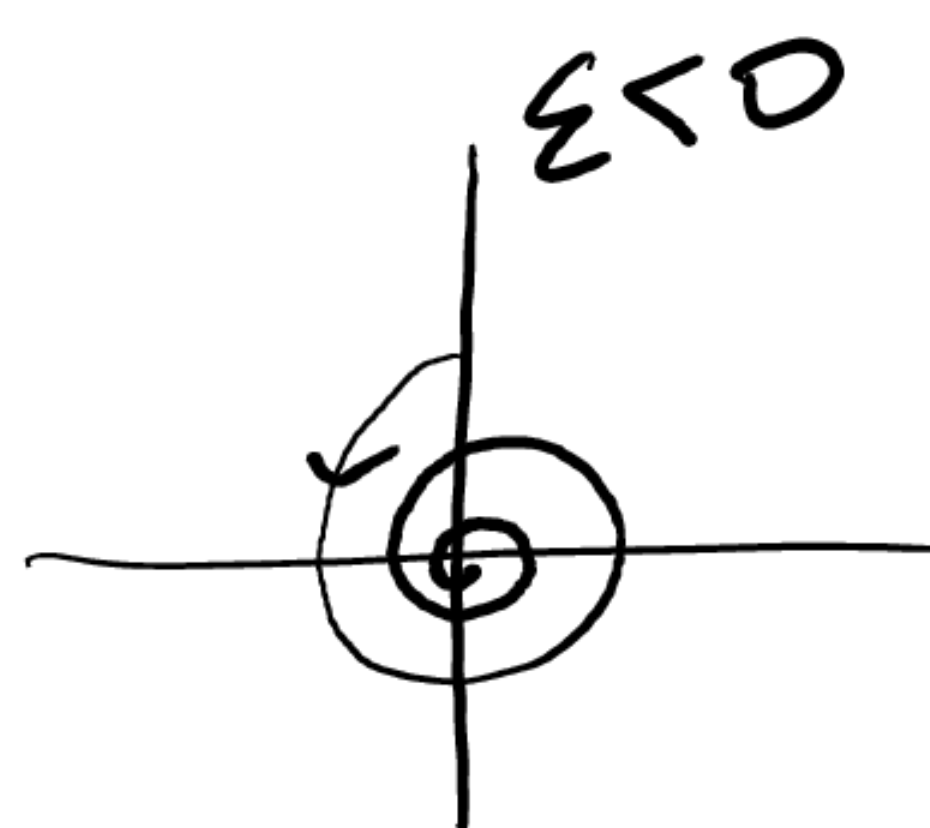
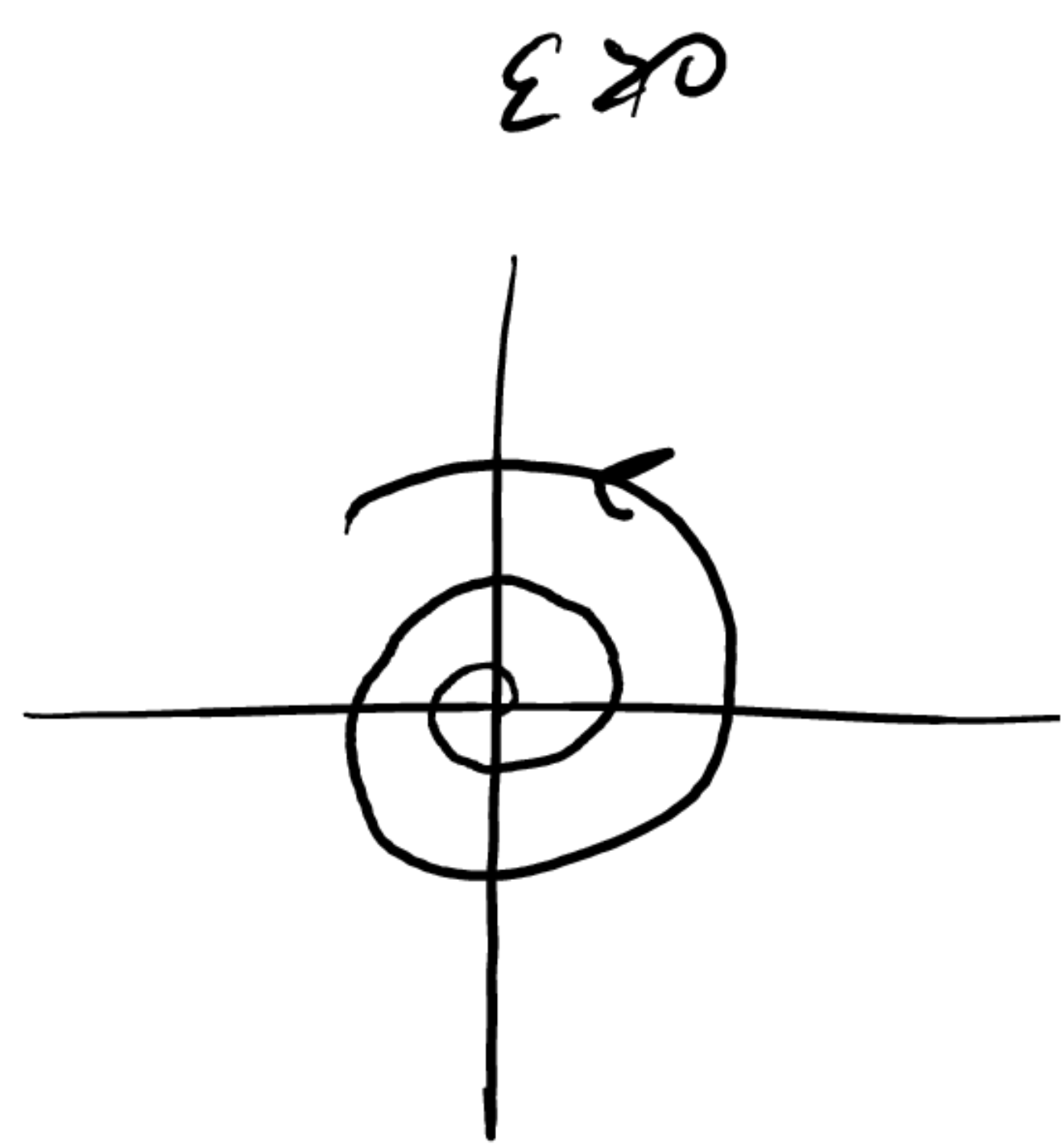
$$y' = r' \sin \theta + r \theta' \cos \theta$$

$$\Rightarrow r' = \frac{x' \cos \theta + y' \sin \theta}{1} = \varepsilon r^3$$

$$\theta' = \frac{-x' \sin \theta + y' \cos \theta}{r} = 1$$

For any (r_0, θ_0) ,

$$r(t) \rightarrow \begin{cases} \infty & \text{if } \varepsilon > 0 \\ 0 & \text{if } \varepsilon < 0. \end{cases}$$



So $\bar{x}^*$ is a $\begin{cases} \text{unstable} & \varepsilon > 0 \\ \text{A-stable} & \varepsilon < 0 \end{cases}$ for $\bar{x}(t)$.

While linearized dynamics

$$\bar{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \bar{x}$$

becomes

$$r' = 0$$

$$\theta' = 1$$

(Verify)

Polar Coordinates

Let $(r, \theta) \in (0, \infty) \times \underbrace{[-\pi, \pi]}_{\mathbb{T}} =: D$

with mapping $H(r, \theta) = \begin{bmatrix} r \cos \theta \\ r \sin \theta \end{bmatrix} \in \mathbb{R}^2 \setminus \{0\}$

Metric: $d_D((r_1, \theta_1), (r_2, \theta_2)) = |H(r_1, \theta_1) - H(r_2, \theta_2)|$

H is a homeomorphism between

(D, d_D) and $\mathbb{R}^2 \setminus \{0\}$

and by deriving dynamics for

$$\dot{\mathbf{r}}' = \begin{bmatrix} \dot{r}' \\ \dot{\theta}' \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

we get a conjugacy between

$$\underbrace{H\left(\mathbf{r}'(t_j; \begin{bmatrix} r_0 \\ \theta_0 \end{bmatrix})\right)}_{\in D} = \underbrace{\mathbf{X}(t_j; H(\begin{bmatrix} r_0 \\ \theta_0 \end{bmatrix}))}_{\in \mathbb{R}^2 \setminus \{0\}}$$

To study $\Sigma(t; \Sigma_0)$ for Σ_0 near $\Sigma^* = 0$,
 study instead conjugate dynamics
 $\Sigma(t; [\begin{smallmatrix} r_0 \\ \theta_0 \end{smallmatrix}])$ for $[\begin{smallmatrix} r_0 \\ \theta_0 \end{smallmatrix}] = H^{-1}(\Sigma_0)$.

Note $|\Sigma(t)| = r(t)$, so $r(t)$ tells us about
 stability properties near 0:

Σ^* A-stable $\Leftrightarrow r(t; [\begin{smallmatrix} r_0 \\ \theta_0 \end{smallmatrix}]) \xrightarrow{t \rightarrow \infty} 0$
 a sink
 $\forall (r_0, \theta_0) \in (0, \delta) \times [-\pi, \pi)$
 for some $\delta > 0$,

Σ^* circular center $\Leftrightarrow r'(t) = 0$ & $\theta' = c \neq 0$

Σ^* source $\Leftrightarrow$ there exists a $\delta_1 > 0$ s.t.
 for any sufficiently small $r_0 \in (0, \delta_1)$
 and $\theta_0 \in [-\pi, \pi)$,
 $r(t; [\begin{smallmatrix} r_0 \\ \theta_0 \end{smallmatrix}]) > \delta_1$ for some $t > 0$.