

Exercises Weeks 11-13

7.1 Write first few terms of Picard iteration, find explicit solutions if possible, and maximal domain of definition (T_-, T_+) .

$$\begin{aligned} a) \quad x' &= x + 2 = f(x) \\ x(0) &= 2 \end{aligned}$$

$$x_0(t) = 2$$

$$\begin{aligned} x_1(t) &= 2 + \int_0^t f(x_0(s)) ds = 2 + \int_0^t 4 ds \\ &= 2 + 4t \end{aligned}$$

$$\begin{aligned} x_2(t) &= 2 + \int_0^t f(2+4s) ds \\ &= 2 + \int_0^t (2+4s+2) ds = 2 + 4t + 2t^2 \end{aligned}$$

$$x_3(t) = 2 + 4t + 2t^2 + \frac{2}{3}t^3 \quad \dots$$

Unique explicit solution: By formula $x' = \underbrace{1}_1 x + \underbrace{2}_2$

$$\begin{aligned} x(t) &= e^{At} \left(x_0 + \int_0^t e^{-As} g(s) ds \right) \\ &= e^t (2 + 2(1 - e^{-t})) = 4e^t - 2. \end{aligned}$$

which is a global solution $T_{\pm} = \pm \infty$.

$$b) \quad x' = x^{4/3} = f(x)$$

$$x(0) = 0$$

Picard iteration

$$x_0(t) = 0$$

$$x_1(t) = \int_0^t f(x_0(s)) ds = 0 = x_0(t)$$

Induction leads to $x_k(t) = 0 \quad \forall k$.

Explicit solution

$$x(t) = 0 \quad \text{valid for } t \in \mathbb{R}.$$

suppose there exists another possibly local solution $y \in C^1([-a, a], [0, \infty))$

for some $a > 0$. Then $y \geq 0$ and $y' \geq 0$

$\Rightarrow y(t) = 0 \quad \forall t \leq 0$, so $\exists t_0 \in (0, a)$ s.t.

$y(t_0) = y_0 > 0$ (otherwise $y = x$ on $[-a, a]$)

But by P-L thm, $y' = y^{4/3}$

$$y(t_0) = y_0$$

has unique solution

$$C = \gamma_0^{-\frac{1}{3}} + t_0$$

$$\gamma^{-\frac{1}{3}} = \frac{C-t}{3}$$

$$\gamma(t) = \frac{27}{(C-t)^3} \quad \frac{\gamma(t_0) = \gamma_0}{\frac{27}{(3\gamma_0^{-\frac{1}{3}} + t_0 - t)^3}}$$

$$\text{on } (T_-, T_+) = (-\infty, 3\gamma_0^{-\frac{1}{3}} + t_0).$$

$$\text{Contradiction } \gamma(0) = \frac{27}{(3\gamma_0^{-\frac{1}{3}} + t_0)^3} \neq 0.$$

Hence, $x(t) = 0$ on $\mathbb{R}$ is unique solution.

$$c) \quad x' = x^{4/3}$$

$$x(0) = 1$$

$$x_0(t) = 1$$

$$x_1(t) = 1 + \int_0^t 1^{4/3} ds = 1+t$$

$$\begin{aligned} x_2(t) &= 1 + \int_0^t (1+s)^{4/3} ds = 1 + \left[\frac{(1+s)^{7/3}}{7/3} \right]_0^t \\ &= \frac{4}{7} + \frac{3}{7} (1+t)^{7/3} \end{aligned}$$

by previous exercise, unique solution:

$$x(t) = \frac{27}{(3-t)^3} \quad \text{on } (T_-, T_+) = (-\infty, 3).$$

7.3 Derive Taylor series for $\sin 2t$ by Picard iteration of first-order system of $x'' = -4x$, $x(0) = 0$, $x'(0) = 2$.

Sol $X_1 = x$
 $X_2 = x'$

$$\underline{X}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \underline{X}, \quad \underline{X}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Picard

$$\underline{X}_0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\underline{X}_1(t) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \underline{X}_0 \, ds = \begin{bmatrix} 2t \\ 2 \end{bmatrix}$$

$$\underline{X}_2(t) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \int_0^t \begin{bmatrix} 2 \\ -8s \end{bmatrix} ds = \begin{bmatrix} 2t \\ 2 - 4t^2 \end{bmatrix}$$

$$\Sigma_3(t) = \begin{bmatrix} 2t - \frac{(2t)^3}{3!} \\ 2 - 4t^2 \end{bmatrix}$$

... Write $\Sigma_k = (x_k^1, x_k^2)$. Then

$$x_k^1(t) \rightarrow \sum_{k=0}^{\infty} (-1)^k \frac{(2t)^{2k+1}}{(2k+1)!}$$

we also know by P-L that

$$x_{\infty}^1(t) = x(t) = \sin(2t), \text{ where}$$

last equality follows by solving problem.

7.6 Discuss existence and uniqueness of solutions of

$$x' = x^a$$

$$x(0) = 0$$

for $a > 0$.

One solution for any $a > 0$

$$X(t) = 0 \quad \forall t \in \mathbb{R}, \text{ as then}$$

$$X'(t) = 0 = X(t)^a \quad \text{ok.}$$

Is it unique?

If $a=1$, then yes as we know
unique sol is $X(t) = X_0 e^t = 0$.

If $a > 1$, then yes, as suppose not.

Then $\exists Y(t)$ s.t. $Y(t_0) = Y_0 > 0$ for
some $t_0 > 0$. (Problem only admits
non-negative solutions)

Solved locally, P-L Then gives unique
solution

$$\frac{1}{1-a} Y^{1-a} = t - t_0 + C$$

$$Y(t) = \left(Y_0^{a-1} + (a-1)(t_0 - t) \right)^{\frac{1}{1-a}}$$

for $(T_-, T_+) = [-\infty, \frac{\gamma_0^{a-1}}{a-1} + t_0]$,

contradicting $X(0) = Y(0)$ as $Y(0) \neq 0$.

Last, if $a < 1$, then for each $T \geq 0$

$$X(t) = \begin{cases} 0 & t \leq T \\ (1-a)^{\frac{1}{1-a}} (t-T)^{\frac{1}{1-a}} & t > T \end{cases}$$

is also a solution. (as $x'(T-) = x'(T+)$
and for $t > T$, $x'(t) = (1-a)^{\frac{a}{1-a}} (t-T)^{\frac{a}{1-a}} = (x(t))^a$)

How to find these extra solutions?
Ansatz $x(t) = \frac{(t-T)^{\beta}}{C}$ for some $\beta > 1$,
& $C > 0$

$$x' = x^a \Rightarrow \frac{\beta}{C} (t-T)^{\beta-1} = \frac{(t-T)^{\beta a}}{C^a}$$

$$\left. \begin{aligned} \text{Eq. } \beta-1 &= \beta a \\ \frac{\beta}{C} &= C^{-a} \end{aligned} \right\} \Rightarrow \begin{aligned} \beta &= \frac{1}{1-a} \\ C &= (1-a)^{-1/(1-a)} \end{aligned}$$

17.4 For each $f(x)$, find a Lipschitz constant in the given domain, or show that f is not globally Lipschitz.

a) $f(x) = |x| \quad x \in \mathbb{R}.$

For $x, y \in \mathbb{R}$

$$f(x) - f(y) = |x| - |y| \leq |x - y| + \cancel{|y| - |x|}$$

Similarly, $f(y) - f(x) \leq |x - y|$

$$\Rightarrow |f(x) - f(y)| \leq L|x - y| \text{ with}$$

Lipschitz const $L = 1$.

b) $f(x) = x^{1/3} \quad x \in [-1, 1] = D$

Function is not even well-defined for $x < 0$, so it is not Lipschitz on D .

Consider rather $D = [0, 1)$. Is it globally Lipschitz there?

Consider $\gamma > 0$ and

$$|f(\gamma) - f(0)| = \gamma^{1/3}$$

As $L > 0$ must satisfy

$$|f(\gamma) - f(0)| \leq L |\gamma| \quad \forall \gamma > 0$$

$$L \geq \sup_{\gamma \in (0,1)} \frac{|f(\gamma) - f(0)|}{|\gamma|}$$

$$= \sup_{\gamma \in (0,1)} |\gamma|^{-2/3} = \infty$$

so f is not Lipschitz on $[0,1)$.

$$d) F(\underline{x}) = \begin{bmatrix} x_1 + 2x_2 \\ -x_2 \end{bmatrix} \quad \underline{x} \in \mathbb{R}^2$$

$$\begin{aligned} |F(\underline{x}) - F(\underline{y})| &= \left| \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} (\underline{x} - \underline{y}) \right| \\ &\leq \left\| \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \right\|_2 |\underline{x} - \underline{y}| \end{aligned}$$

F is Lipschitz with $L = \left\| \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \right\|_2$.

$$e) f(x, y) = \frac{xy}{1+x^2+y^2} \quad \underbrace{x^2+y^2 \leq 4}_{\overline{B_2(0)}}$$

write $f(\underline{x}) = \frac{x_1 x_2}{1+|\underline{x}|^2}$ (scalar-valued function)

For $\underline{x}, \underline{y} \in \overline{B_2(0)}$, let

$$U(s) = \underline{y} + s(\underline{x} - \underline{y}) \quad s \in [0, 1].$$

Then

$$|f(\underline{y}) - f(\underline{x})| = \left| \int_0^1 \frac{d}{ds} f(U(s)) ds \right|$$

$$= \left| \int_0^1 Df(U(s)) U'(s) ds \right|$$

$$\leq L |\underline{x} - \underline{y}|$$

where $L = \max_{\underline{x} \in \overline{B_2(0)}} |Df(\underline{x})| < \infty$ and

$$Df(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} \text{ (is continuous}$$

and thus has $\max L < \infty$ on compact set $\overline{B}_2(0)$).

7.7 Let $A \in C(\mathbb{R}, \mathbb{R}^{n \times n})$ and let

$P(t)$ solve $P' = A(t)P$, $P(0) = P_0 \in \mathbb{R}^{n \times n}$.

Show that $\det P = \det(P_0) \exp\left(\int_0^t \text{Tr } A(s) ds\right)$

Sol By the hints,

$$\det(I + \Delta t A(t)) = 1 + \Delta t \text{Tr } A(t) + o(\Delta t)$$

and

$$\begin{aligned} \det(P(t + \Delta t)) &= \det((I + \Delta t A)P(t)) + o(\Delta t) \\ &= \det(I + \Delta t A) \det P(t) + o(\Delta t) \\ &= (1 + \Delta t \text{Tr } A(t)) \det P(t) \end{aligned}$$

Hence

$$\frac{d}{dt} \det(P(t)) = \lim_{\Delta t \rightarrow 0} \frac{\det(P(t+\Delta t)) - \det(P(t))}{\Delta t}$$

$$= \text{Tr}(A(t)) \det(P(t))$$

Set $\gamma(t) := \det(P(t))$. Then

$$\gamma'(t) = \text{Tr}(A(t)) \gamma \quad \text{with}$$

unique solution

$$\gamma(t) = \underbrace{\det(P_0)}_{\gamma_0} e^{\int_0^t \text{Tr}(A(s)) ds}$$

(Exercise 1.9)

Consequence:

If P_0 is linearly independent, then set of solutions

$$P'_k(t) = A(t) P_k(t) \quad P_k = \begin{bmatrix} p_{1k} \\ \vdots \\ p_{nk} \end{bmatrix} \quad k=1, \dots, n$$

are a set of linearly independent solutions for all t

$$\left(\text{as } \det(P(t)) = \underbrace{\det(P_0)}_{\neq 0} \cdot \underbrace{e^{\int_0^t \text{Tr}(A(s)) ds}}_{\neq 0} \right)$$

And the unique solution of

$$X' = A(t) X$$

$$X(0) = X_0$$

can be written

$$X(t) = \sum_{k=1}^n c_k P_k(t), \text{ where } C \text{ solves}$$

$$P_0 \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = X_0.$$

17.10 Show that BVP

$$x'' = -x, \quad x(0) = 0, \quad x(\pi) = 1$$

has no solution.

Rewrite as first-order problem

$$\left. \begin{array}{l} x_1 = x \\ x_2 = x' \end{array} \right\} \Rightarrow \underline{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \underline{x}$$

$$(x_1(0) = 0, \quad x_1(\pi) = 1)$$

General solution:

$$\underline{x}(t) = c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

$$x_1(0) = 0 \Rightarrow c_1 = 0$$

$$x_1(\pi) = 1 \Rightarrow c_2 \underbrace{\sin \pi}_{=0} = 1$$

$\Rightarrow$ no solution.

17.11 Consider

$$x'' = -kx \quad \text{for } k > 0$$

$$x(0) = 0, x(\pi) = 1$$

does there now exist a solution?

Sol First order problem

$$\left. \begin{array}{l} x_1 = x \\ x_2 = \frac{x'}{\sqrt{k}} \end{array} \right\} \Rightarrow \begin{array}{l} x_1' = \sqrt{k} x_2 \\ x_2' = -\sqrt{k} x_1 \end{array} \quad \text{i.e.}$$

$$\underline{x}'(t) = \begin{bmatrix} 0 & \sqrt{k} \\ -\sqrt{k} & 0 \end{bmatrix} \underline{x} \quad \text{with general solution}$$

$$\underline{x}(t) = c_1 \begin{bmatrix} \cos \sqrt{k} t \\ -\sin \sqrt{k} t \end{bmatrix} + c_2 \begin{bmatrix} \sin \sqrt{k} t \\ \cos \sqrt{k} t \end{bmatrix}$$

$$x_1(0) = 0 \Rightarrow c_1 = 0$$

$$x_1(\pi) = 1 \Rightarrow c_2 \sin \sqrt{k} \pi = 1$$

Has a solution when $\sqrt{k} \in \mathbb{N}$.

