

Recall

For $\underline{x}' = F(\underline{x})$ in $\mathbb{R}^n$, an EP $\underline{x}^*$ is

- stable if for any $\delta > 0$, $\exists \epsilon \in (0, \delta)$ s.t. $|\underline{x}(t; \underline{x}_0) - \underline{x}^*| < \delta \quad \forall (\underline{x}_0 \in B_\epsilon(\underline{x}^*) \text{ \& } t \geq 0)$

- A-stable if $\underline{x}^*$ is stable and $\exists \delta > 0$ s.t.

$$\lim_{t \rightarrow \infty} |\underline{x}(t; \underline{x}_0) - \underline{x}^*| = 0 \quad \forall \underline{x}_0 \in B_\delta(\underline{x}^*)$$

- Unstable if it is not stable.

(There exist a $\delta > 0$, s.t. one can find $\underline{x}_0$ arbitrarily close to $\underline{x}^*$ satisfying $|\underline{x}(t; \underline{x}_0) - \underline{x}^*| > \delta$ for some $t > 0$.)

Known results for EP $\bar{x}^*$ of $\bar{x}' = F(\bar{x})$ with $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$.

Thm 1 (linearization thm)

If $DF(\bar{x}^*)$ is hyperbolic ($\operatorname{Re} \lambda \neq 0 \forall \text{ EV}$), then $\bar{x}^*$ for $\bar{x}$ is of same type as $\bar{x}^*$ for $\tilde{\bar{x}}' = DF(\bar{x}^*)(\tilde{\bar{x}} - \bar{x}^*)$.

Consequences:

- If $\operatorname{Re}(\lambda) < 0$ for all EV of $DF(\bar{x}^*)$, then $\bar{x}^*$ is a A-stable (a sink)
- If $\operatorname{Re}(\lambda) < 0$ for at least one EV, and $\operatorname{Re}(\lambda) > 0$ for at least one EV, and $DF(\bar{x}^*)$ is hyperbolic, then $\bar{x}^*$ is a saddle.

Can further show.

If $\operatorname{Re}(\lambda) > 0$ for at least one EV, then $\bar{x}^*$ is unstable.

Note Holds even if $DF(\bar{x}^*)$ is not hyperbolic.

Ex 1 $\underline{X}' = \begin{bmatrix} \sin(x_1) + x_1 x_2 \\ \cos(x_2) x_1 + x_2 \end{bmatrix}$

$\underline{X}^* = 0$ is an EP and

$$DF(\underline{X}) = \begin{bmatrix} \cos(x_1) + x_2 & x_1 \\ \cos(x_2) & 1 - x_1 \sin x_2 \end{bmatrix}$$

$$DF(\underline{X}^*) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = 1 \Rightarrow 0$ is unstable (and a source)

Ex 2

$$\underline{X}' = \begin{bmatrix} x_1^2 \\ e^{x_1} x_2 \end{bmatrix}$$

$$\text{EP: } \underline{X}^* = 0$$

$$DF(\underline{X}) = \begin{bmatrix} 2x_1 & 0 \\ e^{x_1} x_2 & e^{x_1} \end{bmatrix}, \quad DF(0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

0 is unstable.

(Consider $\underline{x}_0 = \begin{bmatrix} 0 \\ d \end{bmatrix}$ for any $d \neq 0$
Then $\underline{x}(t, \underline{x}_0) = \begin{bmatrix} 0 \\ de^t \end{bmatrix}$.)

Saddle points in $\mathbb{R}^2$

If $\det(DF(\underline{x}^*)) = \lambda_1 \lambda_2 < 0$, then
 $\underline{x}^*$ is a saddle.

Recall from lecture 17

$$\underline{x}' = \begin{bmatrix} x_1 + x_2^2 \\ -x_2 \end{bmatrix} \Rightarrow DF(0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Linearised problem

$\tilde{\underline{x}}' = DF(0)\tilde{\underline{x}}$ has stable direction $\underline{e}_2$ /

curve $C = \{(x, y) \mid x = 0\}$: For any

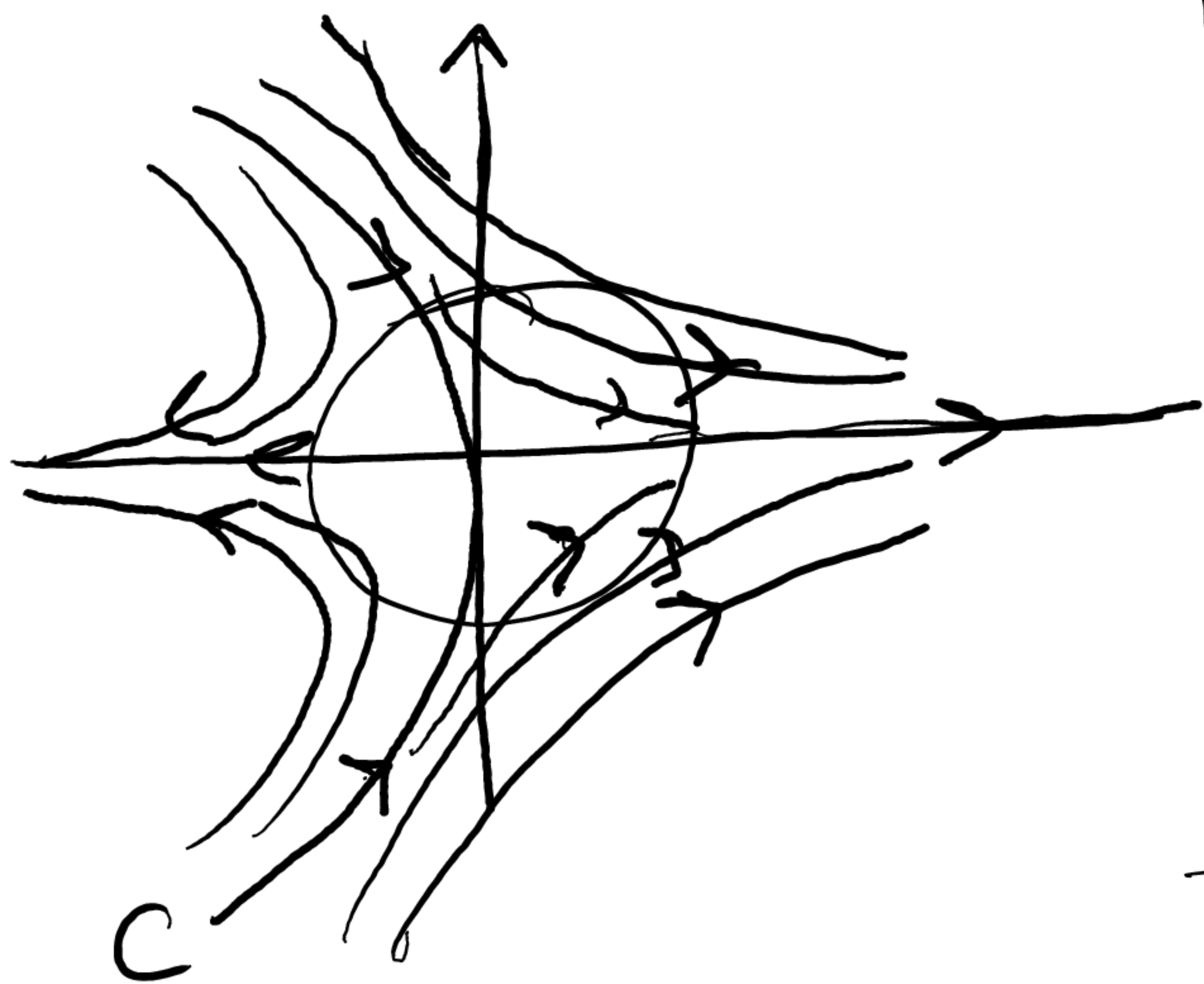
$\underline{x}_0 \in C$, $\tilde{\underline{x}}(t; \underline{x}_0) \in C \forall t \geq 0$ and

$$\lim_{t \rightarrow \infty} \tilde{\underline{x}}(t; \underline{x}_0) = 0.$$

And if $\bar{x}_0 \notin C$, then $|\bar{x}(t; \bar{x}_0)| \rightarrow \infty$.

Similarly, nonlinear problem has a stable curve $C = \{(x, y) \mid x + \frac{1}{3}y^2 = 0\}$.

For any $\bar{x}_0 \in C$, $\bar{x}(t; \bar{x}_0) \in C \forall t \geq 0$ and $\lim_{t \rightarrow \infty} \bar{x}(t; \bar{x}_0) = 0$.



And for any $\bar{x}_0 \notin C$

$$\lim_{t \rightarrow \infty} |\bar{x}(t; \bar{x}_0)| = \infty.$$

This holds locally for any saddle point in $\mathbb{R}^2$.

Theorem (stable curve in $\mathbb{R}^2$)

If $\det(DF(\bar{x}^*)) < 0$, then $\exists \delta > 0$ and $B_\delta(\bar{x}^*)$ and stable curve

$C \in B_\delta(\bar{x}^*)$ s.t. $\bar{x}^* \in C$.

And if $\bar{x}_0 \in B_\delta(\bar{x}^*) \setminus C$, then $\bar{x}(t; \bar{x}_0) \notin B_\delta(\bar{x}^*)$ for some $t > 0$.

Ex 3

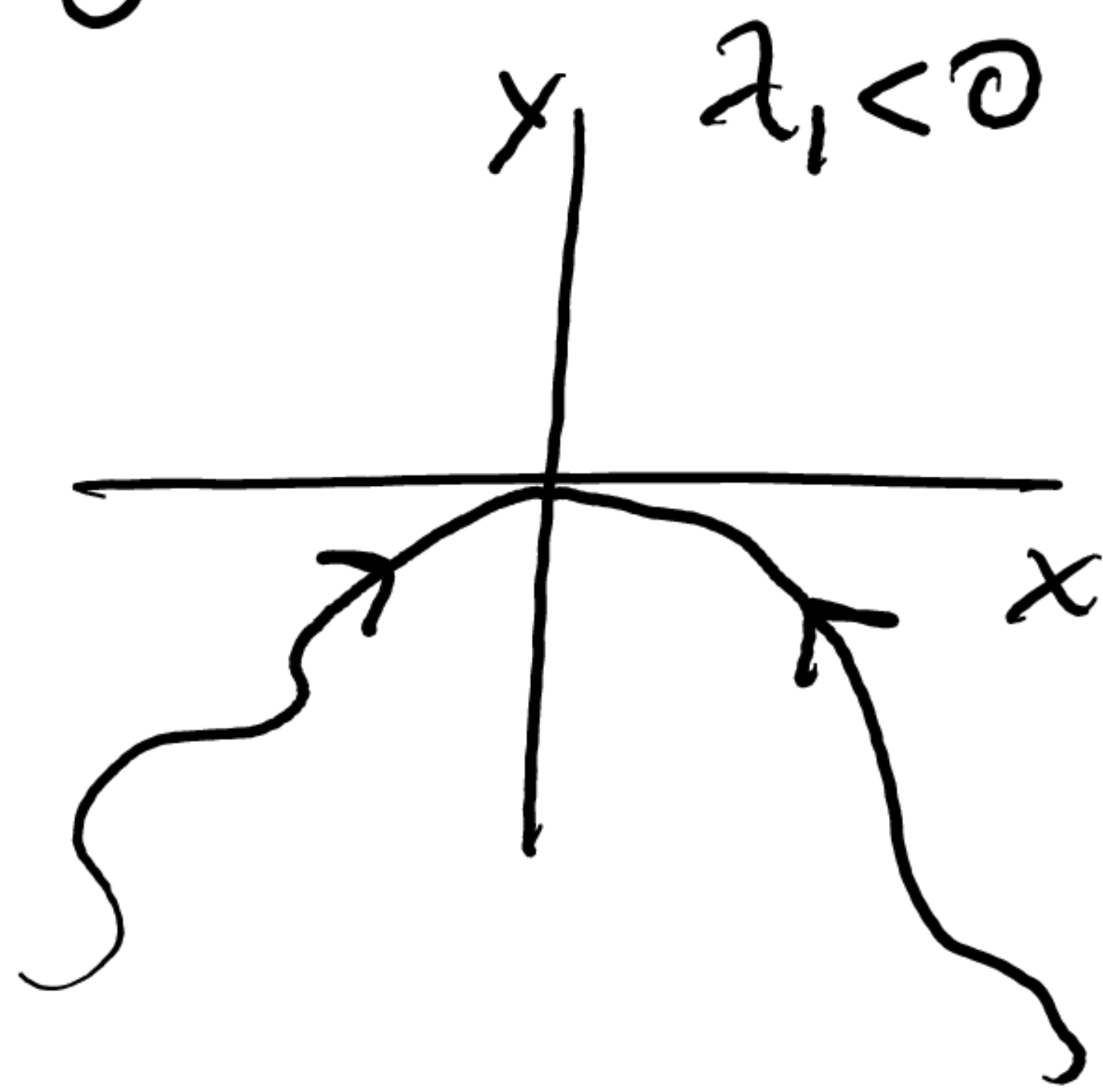
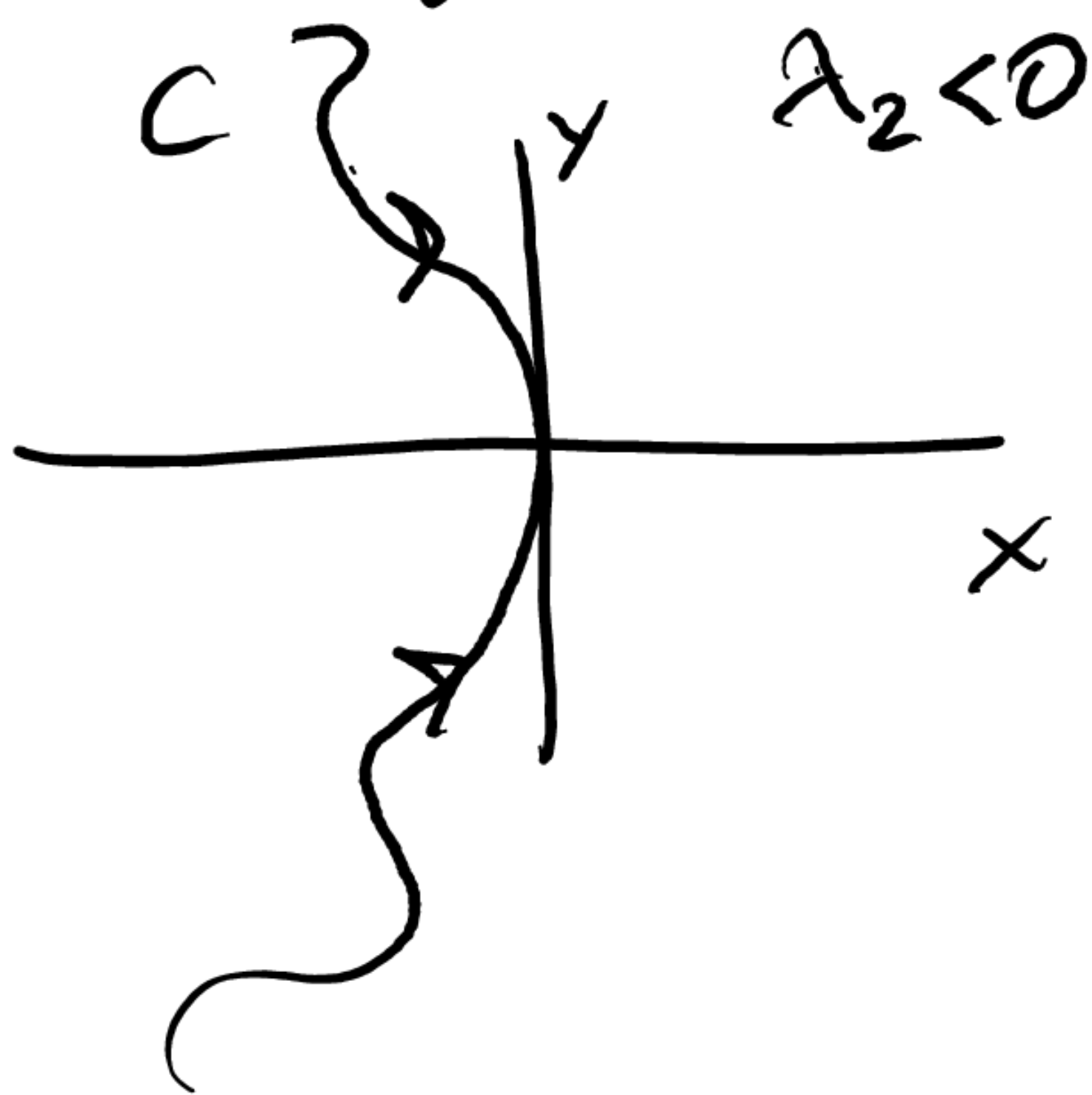
$$x' = \lambda_1 x + f_1(x, y)$$

$$y' = \lambda_2 y + f_2(x, y)$$

with $\lambda_1, \lambda_2 < 0$ and

$$\max_{|X|} (|f_1(X)|, |f_2(X)|) \rightarrow 0 \text{ as } |X| \rightarrow 0$$

always has a locally stable curve C .



At $X^* = 0$, curve C is tangent to

- y -axis if $\lambda_2 < 0$

- x -axis if $\lambda_1 < 0$

(as near X^* , dynamics of $X \approx \tilde{X}$)

Bifurcations in $\mathbb{R}$ and $\mathbb{R}^2$

Consider $\dot{X} = F_a(X)$ (1)

We assume $a \in I \subset \mathbb{R}$, also write
 $G(X, a) = F_a(X)$ and assume G is C^∞ .

For a fixed $a \in I$, problem (1) may have EP $X_1^*(a), \dots, X_m^*(a)$. Both type(s) and number of EP may vary with value a .

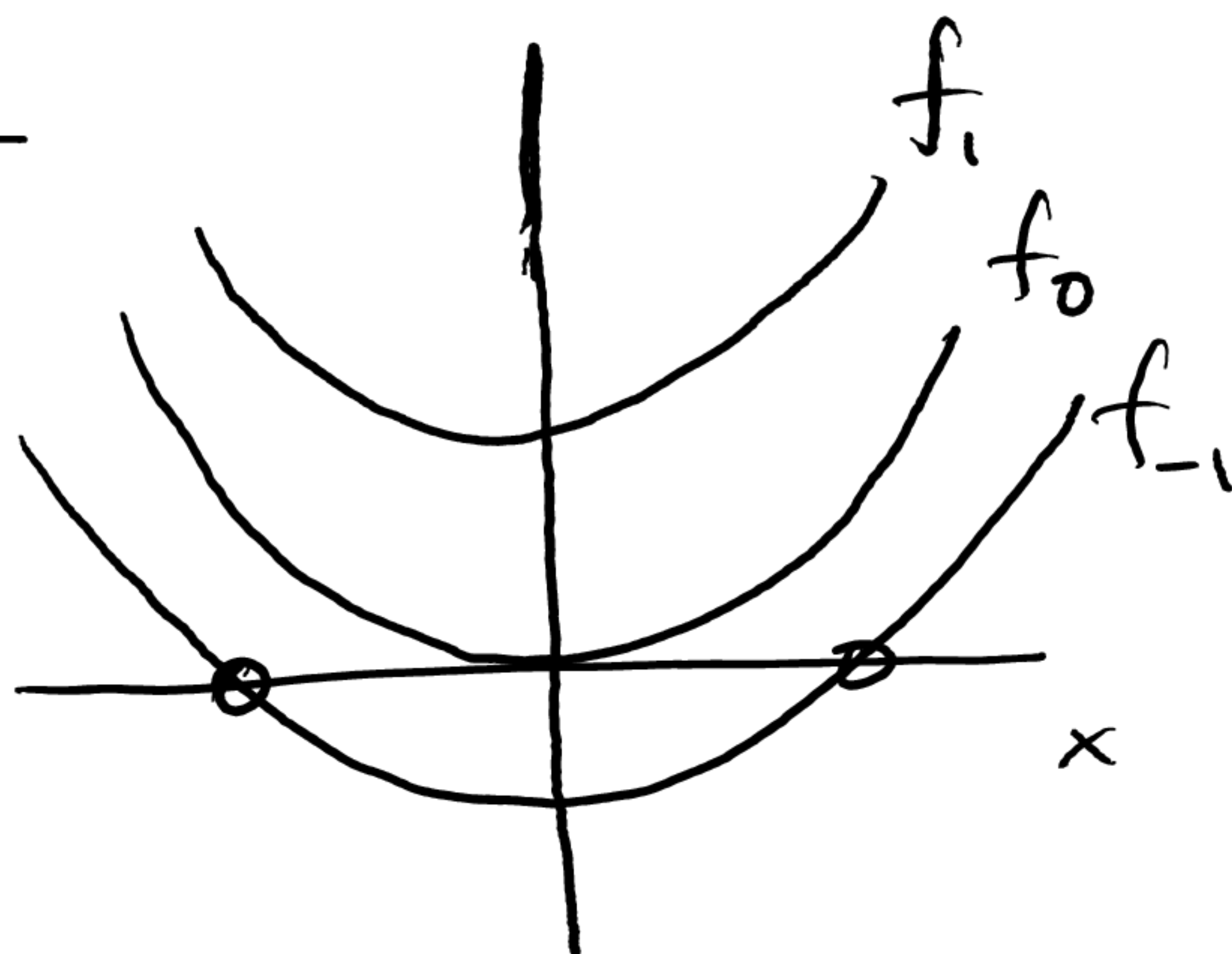
Bifurcation point is a point $a \in I$

where either type(s) or number of EP differs between $a^- = \lim_{\delta \downarrow 0} a - \delta$

and $a^+ = \lim_{\delta \downarrow 0} a + \delta$ for (1).

Examples in $\mathbb{R}$

$$x' = x^2 + a = f_a(x)$$



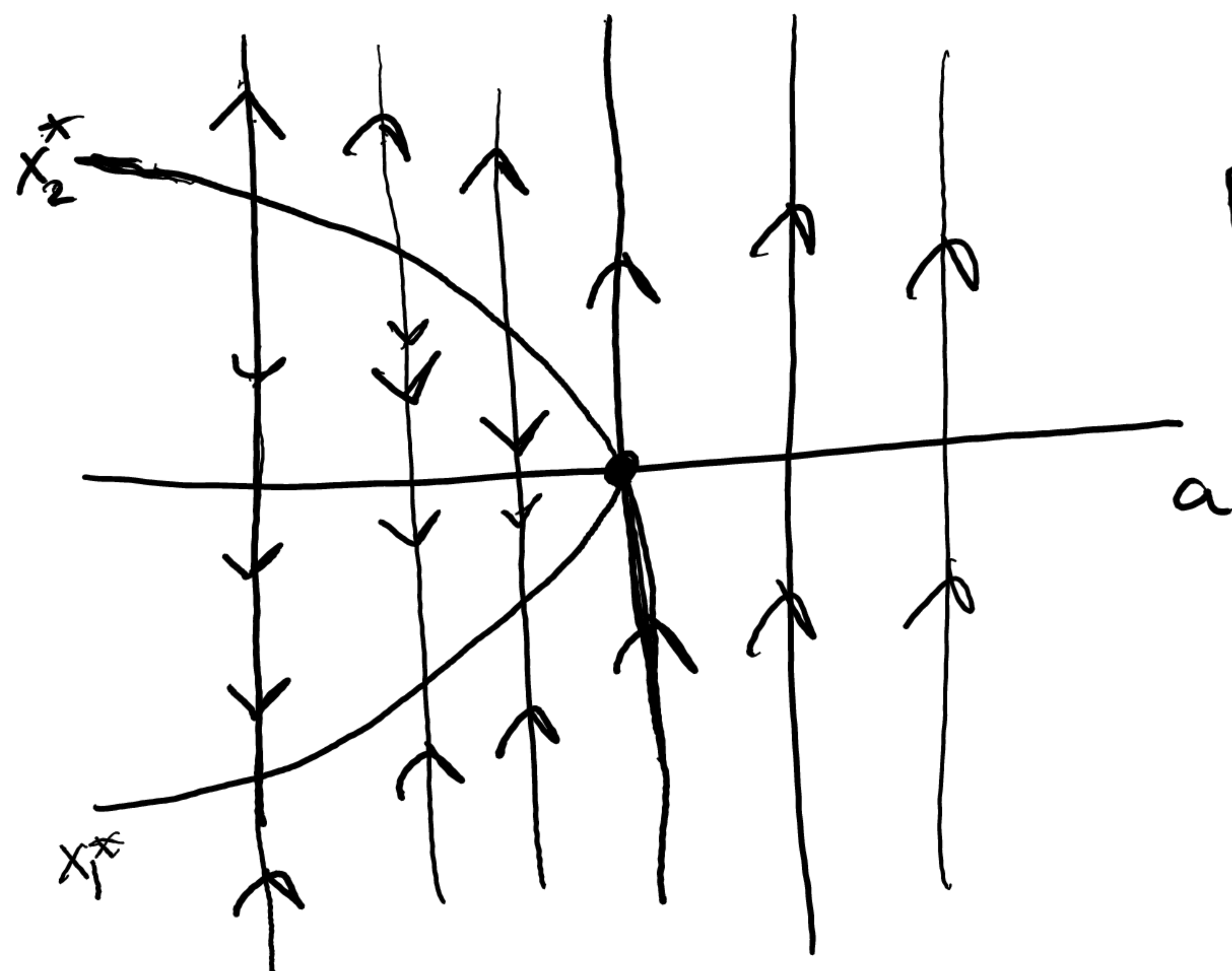
$$EP: f_a(x) = 0$$

- $x_1^*(a) = -\sqrt{-a}$, $x_2^*(a) = \sqrt{-a}$ if $a < 0$

- $x^*(0) = 0$ $a = 0$

- No EP if $a \geq 0$.

Bifurcation diagram



Bifurcation
point: $a = 0$.

- Def 1 A simple saddle-node bifurcation at $a_0 \in I$ has an interval $J \subset \mathbb{R}$ s.t.
- ① Two EP $x_1^*(a), x_2^*(a) \in J$ exist for $a < a_0$ (or $a > a_0$) near a_0 . (One is a sink, the other unstable "1D-saddle")
 - ② One EP $x_0 = x^*(a_0) \in J$ exist for $a = a_0$
 - ③ No EP exist for $a > a_0$ (or $a < a_0$)

Node = source or sink EP

(General saddle-node bifurcation = bifurcation where step ① above involves saddles & nodes.)

Theorem 3 If $x' = f_a(x)$ has a bifurcation at $a = a_0$ with $x_0 = x^*(a_0)$ and 1. $f_{a_0}(x_0) = 0$, 2. $f'_{a_0}(x_0) = \frac{\partial}{\partial x} g(x_0, a_0) = 0$
 3. $\frac{\partial^2}{\partial x^2} g(x_0, a_0) \neq 0$, 4. $\frac{\partial}{\partial a} g(x_0, a_0) \neq 0$,

then it is a simple saddle-node bifurcation.

Proof idea

Near (x_0, a_0) with $g(x_0, a_0) = 0$,
 $g_a(x_0, a_0) \neq 0$ and implicit function theorem
 $\Rightarrow \exists$ smooth function $\hat{a}(x)$ s.t.

$$g(x, \hat{a}(x)) = 0 \text{ for } x \text{ near } x_0.$$

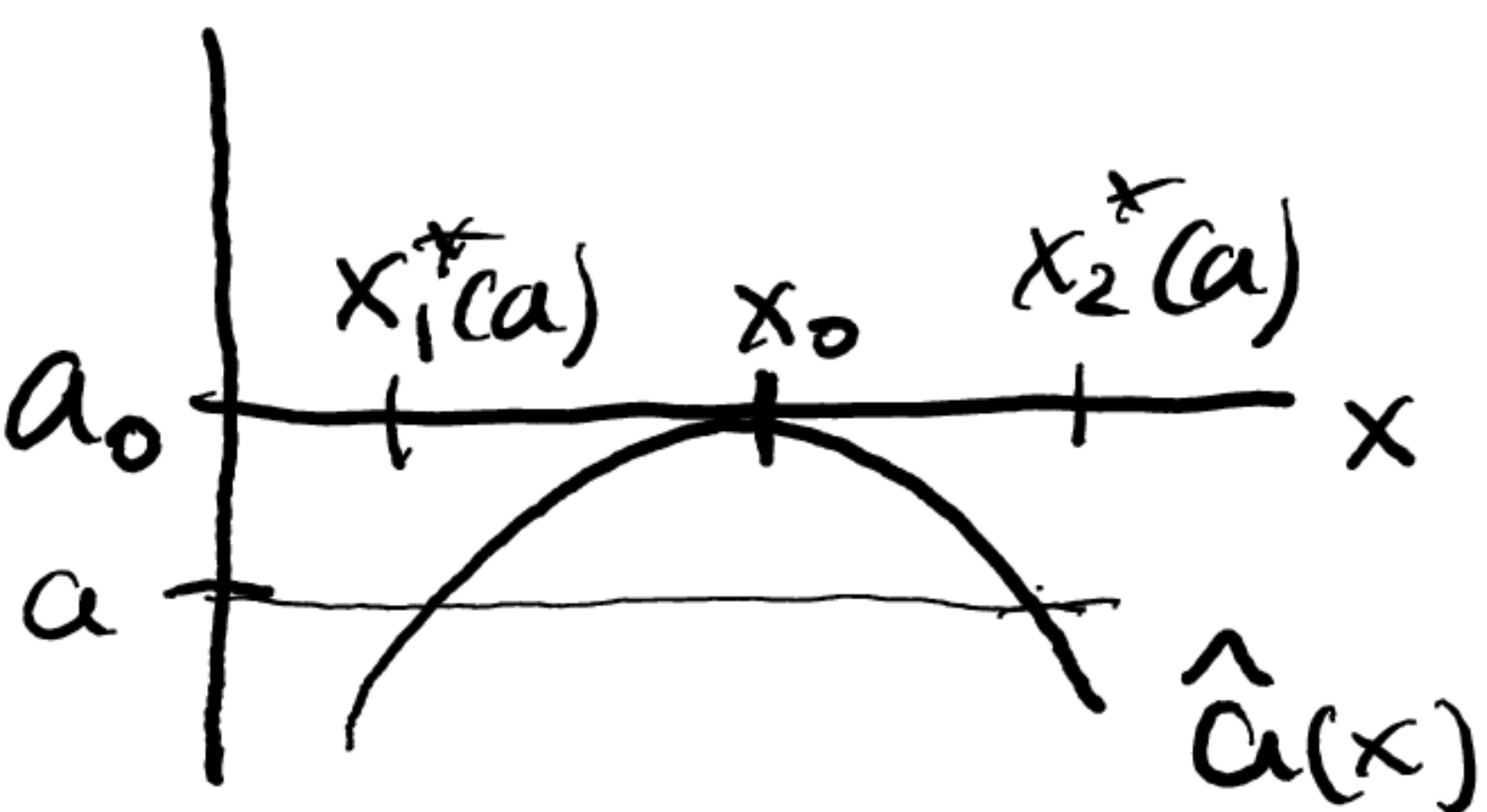
(In particular $\hat{a}(x_0) = a_0$) Further,

$$\frac{d^2}{dx^2} g(x, \hat{a}(x)) = 0 \text{ yields by 2, 3, 4.}$$

that $\hat{a}''(x) \neq 0$, so $\hat{a}$ either strictly
convex or concave near x_0 .

Note near x_0 , x is an EP of $f_{\hat{a}(x)}(\cdot)$
as $f_{\hat{a}(x)}(x) = g(x, \hat{a}(x)) = 0$.

If $\hat{a}$ is concave at x_0 , then $\hat{a}(x_0 \pm \delta) < a_0$



So for
 $a < a_0$ near a_0 , there
exist two points $x_1^*(a) < x_0 < x_2^*(a)$

s.t. $\hat{a}(x_1^*(a)) = \hat{a}(x_2^*(a)) = a$. And these are EP of f_a as $f_a(x_1^*(a)) = f_{\hat{a}(x_1^*)} = g(x_1^*, \hat{a}(x_1^*)) = 0$ etc. Hence point ① holds.

Def 1 points ② & ③ also hold $\square$

Saddle-node bifurcation in 2D

Consider

$$\begin{aligned} x' &= x^2 + a \\ y' &= y \end{aligned}, \quad F(\underline{x}) = \begin{bmatrix} x^2 + a \\ y \end{bmatrix}.$$

For $a < 0$ EP $\underline{x}_1^*(a) = \begin{bmatrix} -\sqrt{-a} \\ 0 \end{bmatrix}$ $\swarrow$ saddle

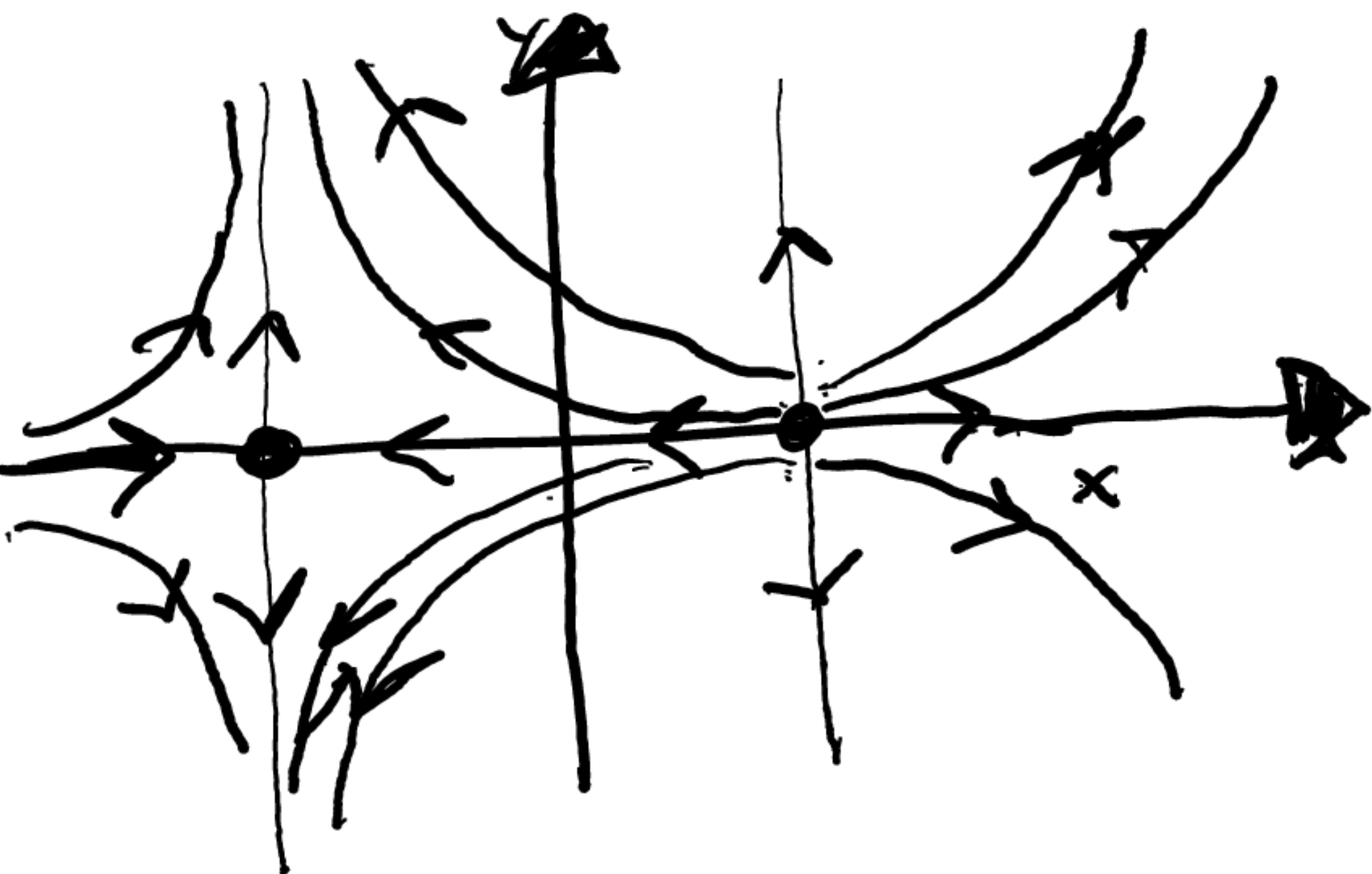
$$\underline{x}_2^*(a) = \begin{bmatrix} \sqrt{-a} \\ 0 \end{bmatrix} \swarrow \text{source}$$

$$DF(\underline{x}) = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix}$$

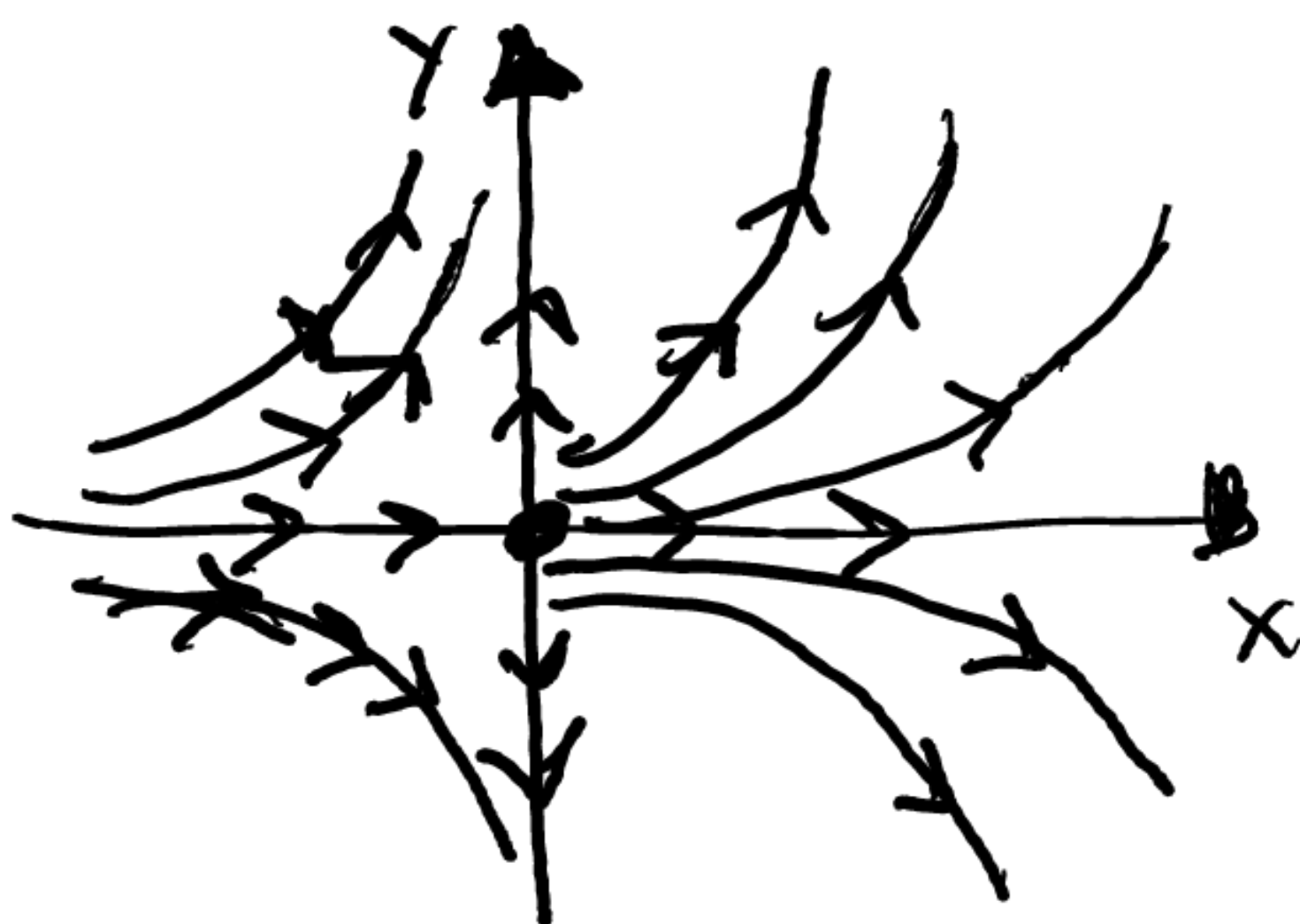
for $a = 0$ EP $\underline{x}^*(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

for $a > 0$ no EP.

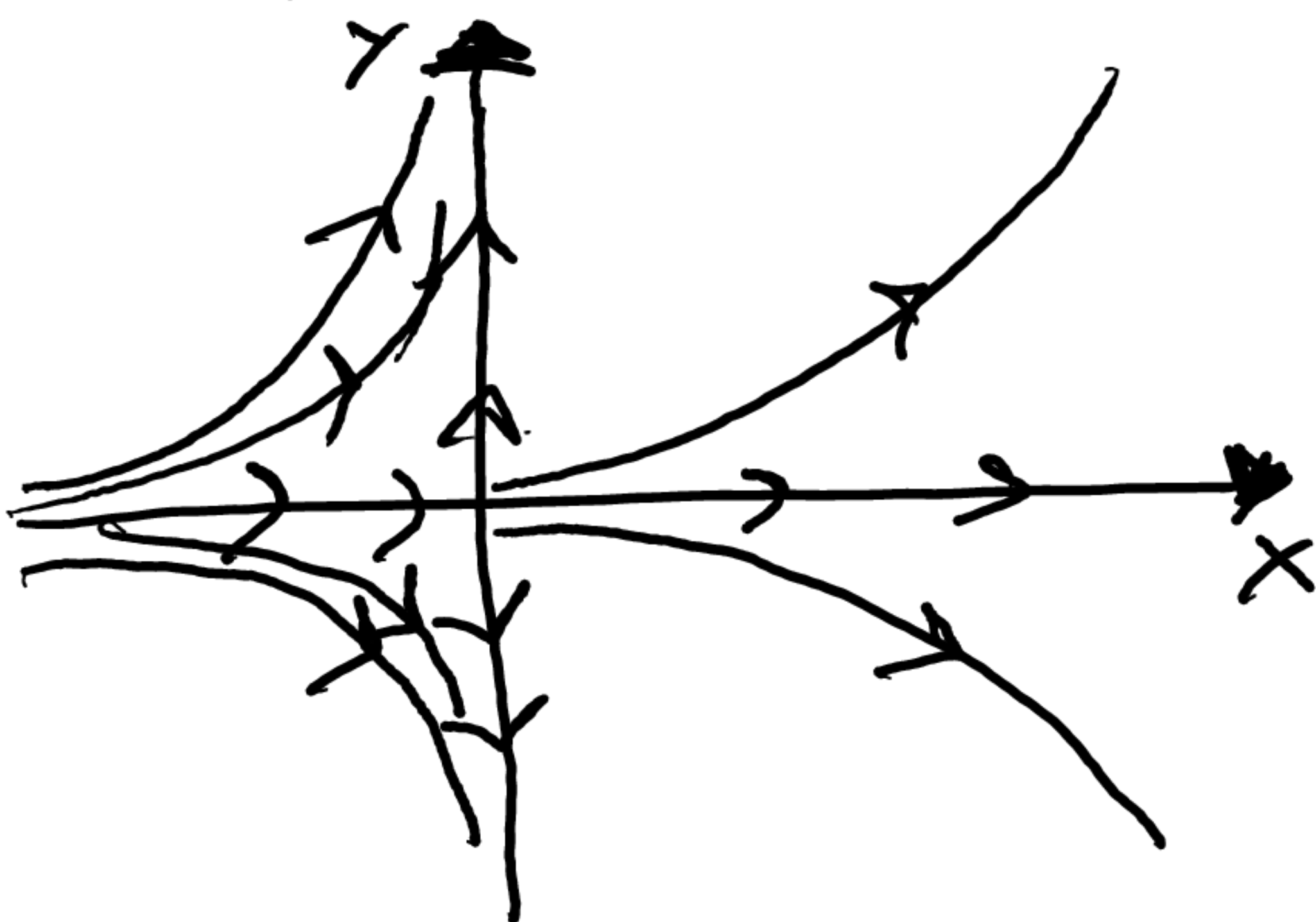
$$a < 0$$



$$a = 0$$



$$a > 0$$



If rather

$$x' = x^2 + a$$

$$y' = -y$$

,

then $\Sigma_1^*(a)$ is a sink and $\Sigma_2^*(a)$ is a saddle.