

Recall $x' = x^2 + a$ for $a \in \mathbb{R}$
 undergoes a saddle-node bifurcation
 at $a_0 = 0$:

sink-source EPs $x_{\pm}^* = \pm \sqrt{-a}$ $a < 0$

one EP $x^* = 0$ $a = 0$

Pitchfork bifurcation

$$x' = ax - x^3 =: f_a(x), \quad a \in \mathbb{R}.$$

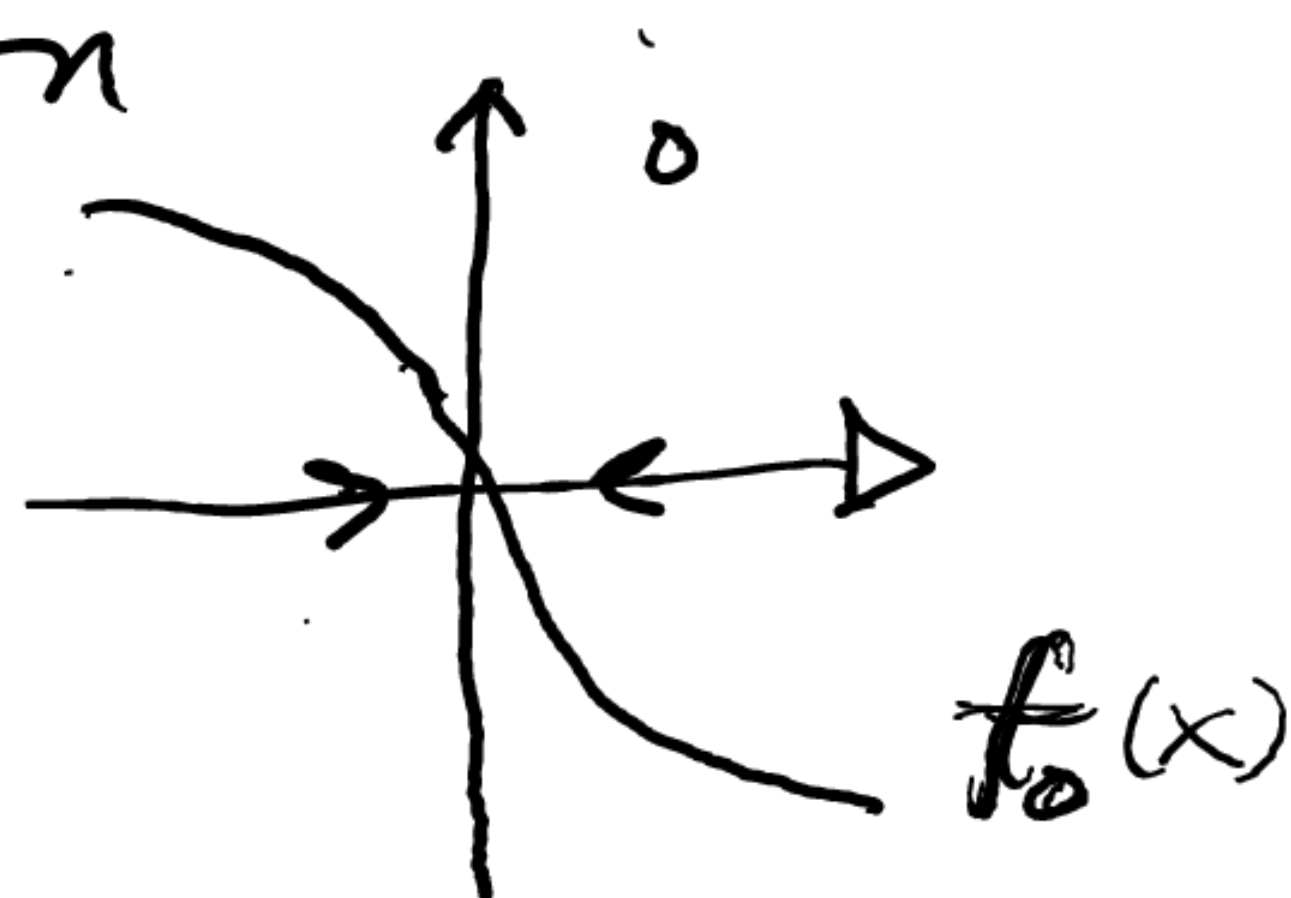
$$\text{EPs} \quad x_1^* = 0 \quad a \leq 0,$$

and additionally $x_2^* = \sqrt{a}$, $x_3^* = -\sqrt{a}$

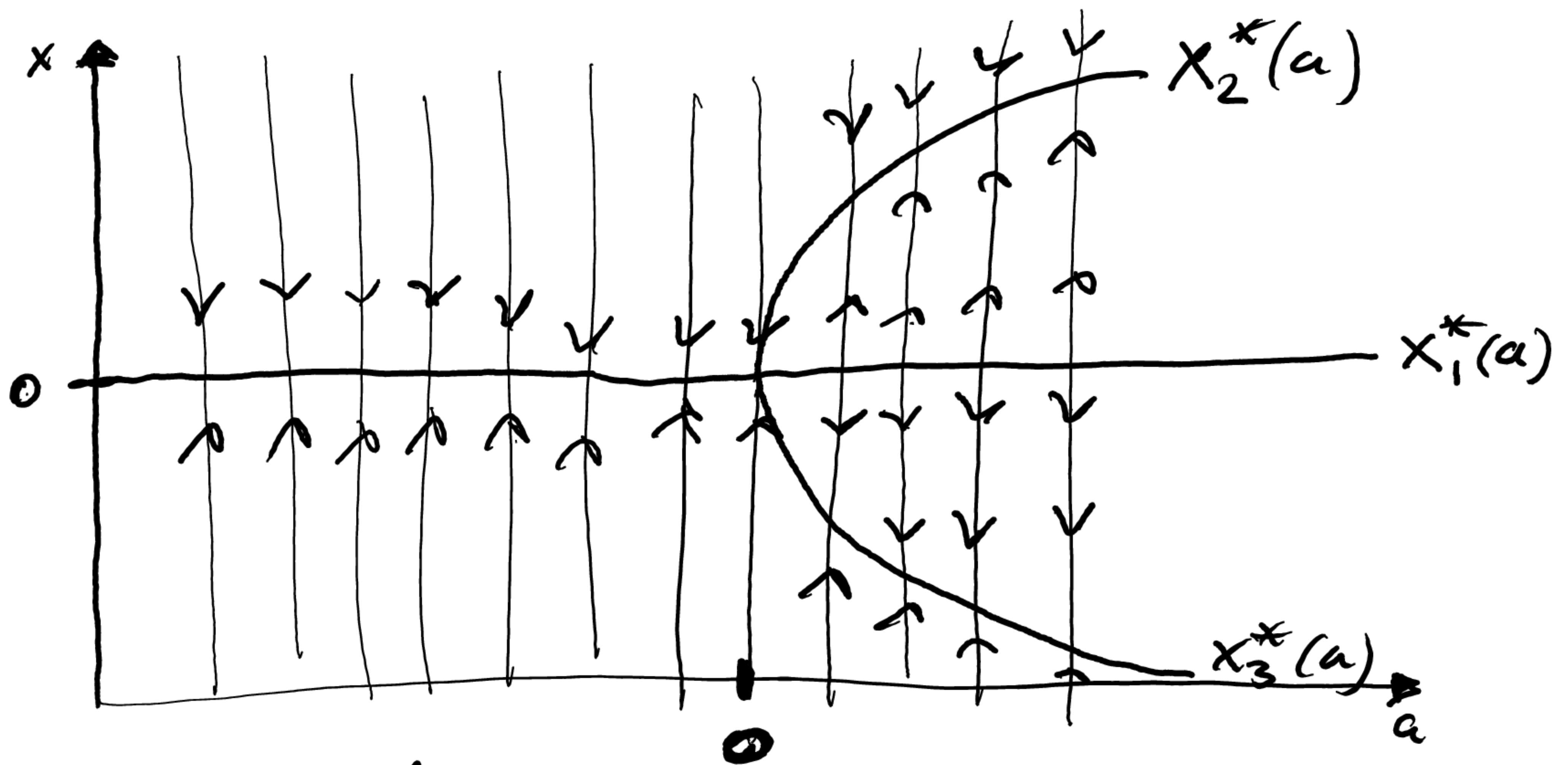
when $a > 0$. Classification

$$f_a'(0) = a \quad \text{and}$$

$$\Rightarrow x_1^*(a) = \begin{cases} \text{sink} & a < 0 \\ \text{A-stable} & a = 0 \\ \text{source} & a > 0 \end{cases}$$



$\Rightarrow x_2^*(a), x_3^*(a)$ are sinks



Application

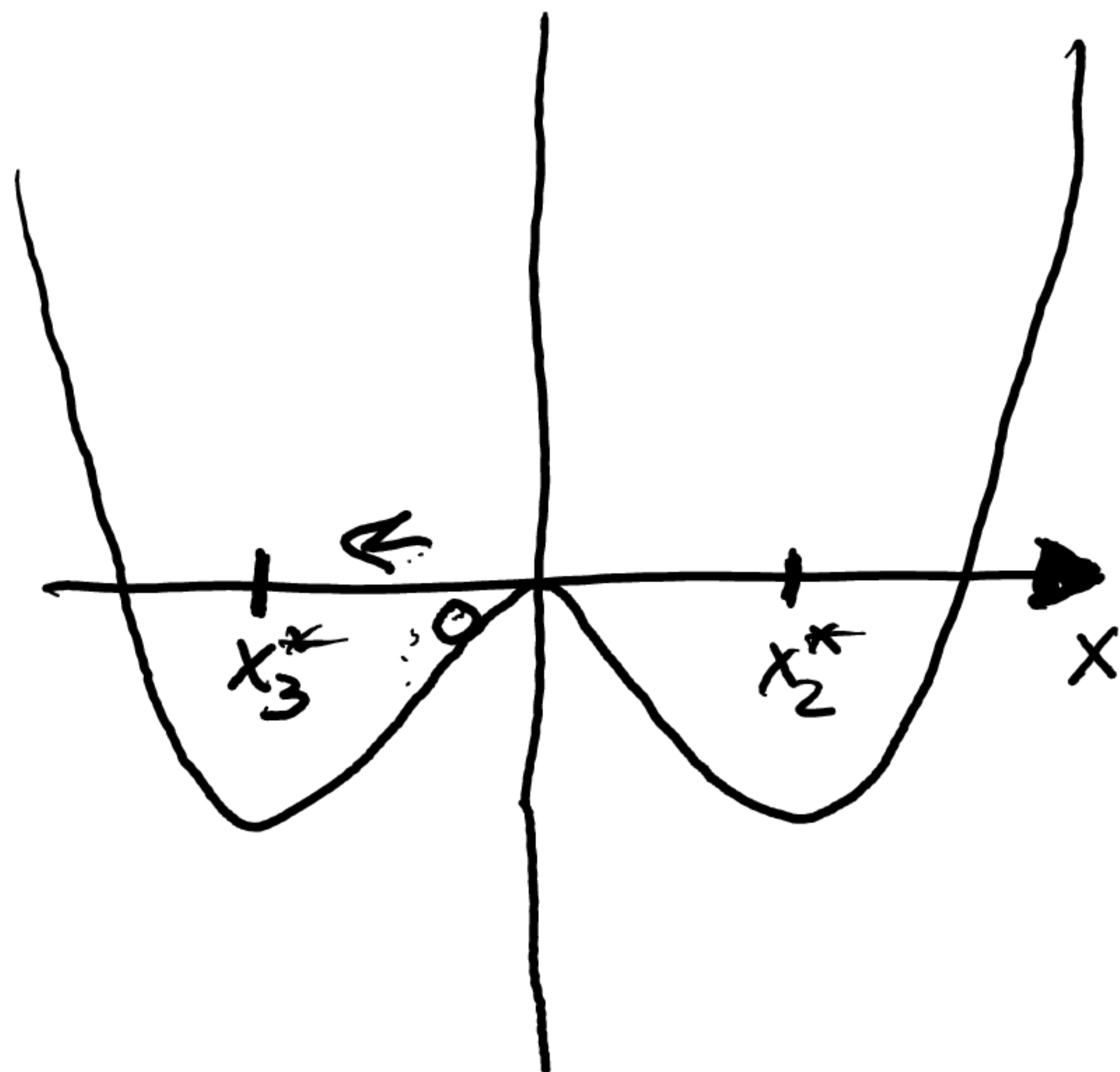
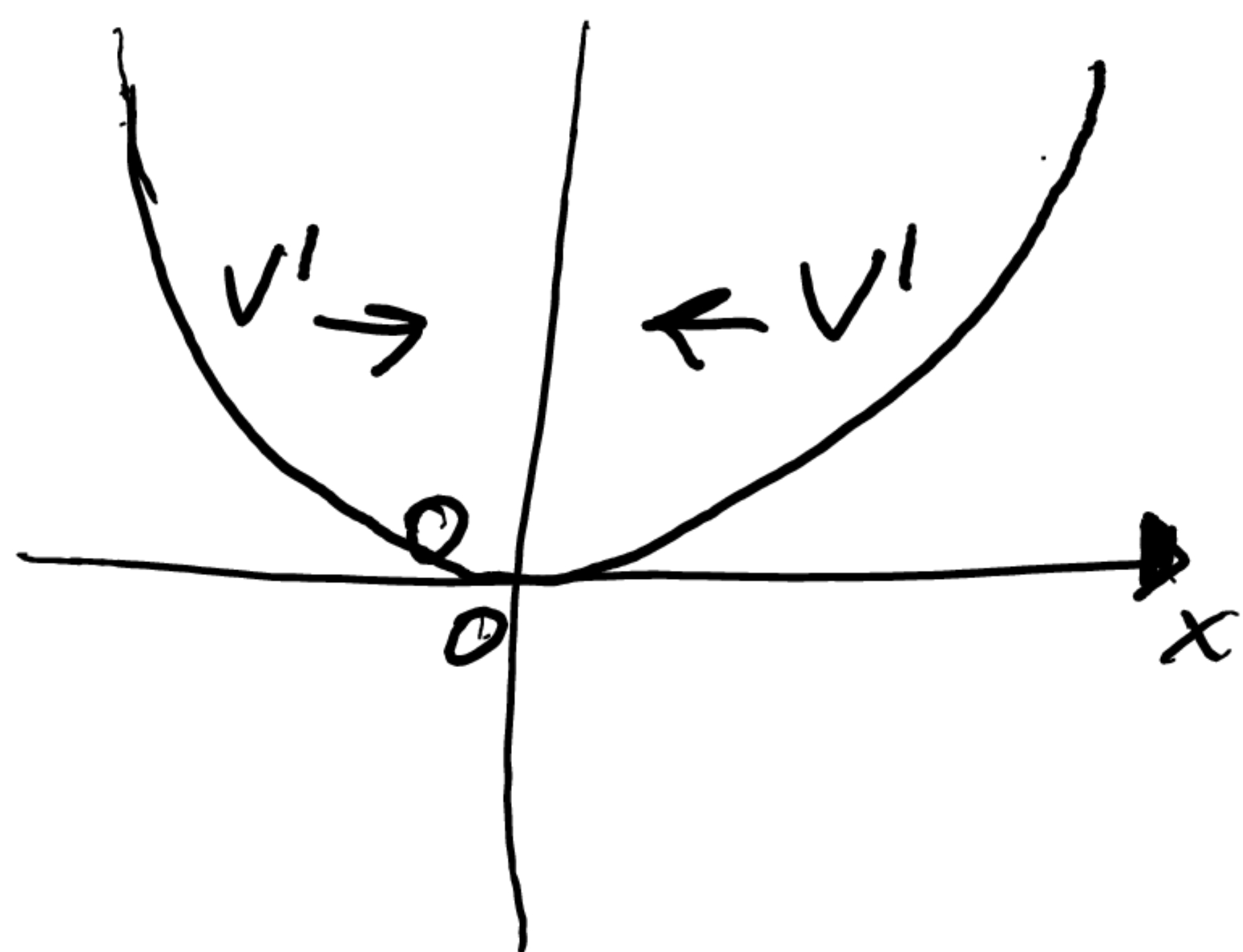
Damped Newtonian dynamics:

if $m \ll 1$, approximate by

Overdamped dynamics / gradient flow

$$x'(t) = -V_a'(x(t))$$

$V_a(x)$ $a < 0$ rise as $V_a(x)$ $a > 0$



(Flows as if affected by gravity)

For $a > 0$, $x(t; x_0) \rightarrow \begin{cases} x_3^* & x_0 < 0 \\ x_1^* & x_0 = 0 \\ x_2^* & x_0 > 0 \end{cases}$

as $t \rightarrow \infty$.

(Parameter a controls where particle will end up for $t \gg 1$.)

Hopf bifurcation example

Transition from stable EP to unstable EP + a limit cycle (periodic solution)

$$\underline{X}' = \begin{bmatrix} ax - y + x(x^2 + y^2) \\ x + ay + y(x^2 + y^2) \end{bmatrix} = \underline{F}_a(\underline{X})$$

EP $\underline{X}^* = 0$. $\underline{X}(t)$ on $\mathbb{R}^2 \setminus \{0\}$

is conjugate to

$$r' = ar - r^3 = f_a(r)$$

$$\theta' = 1$$

on $(r, \theta) \in (0, \infty) \times [-\pi, \pi)$.

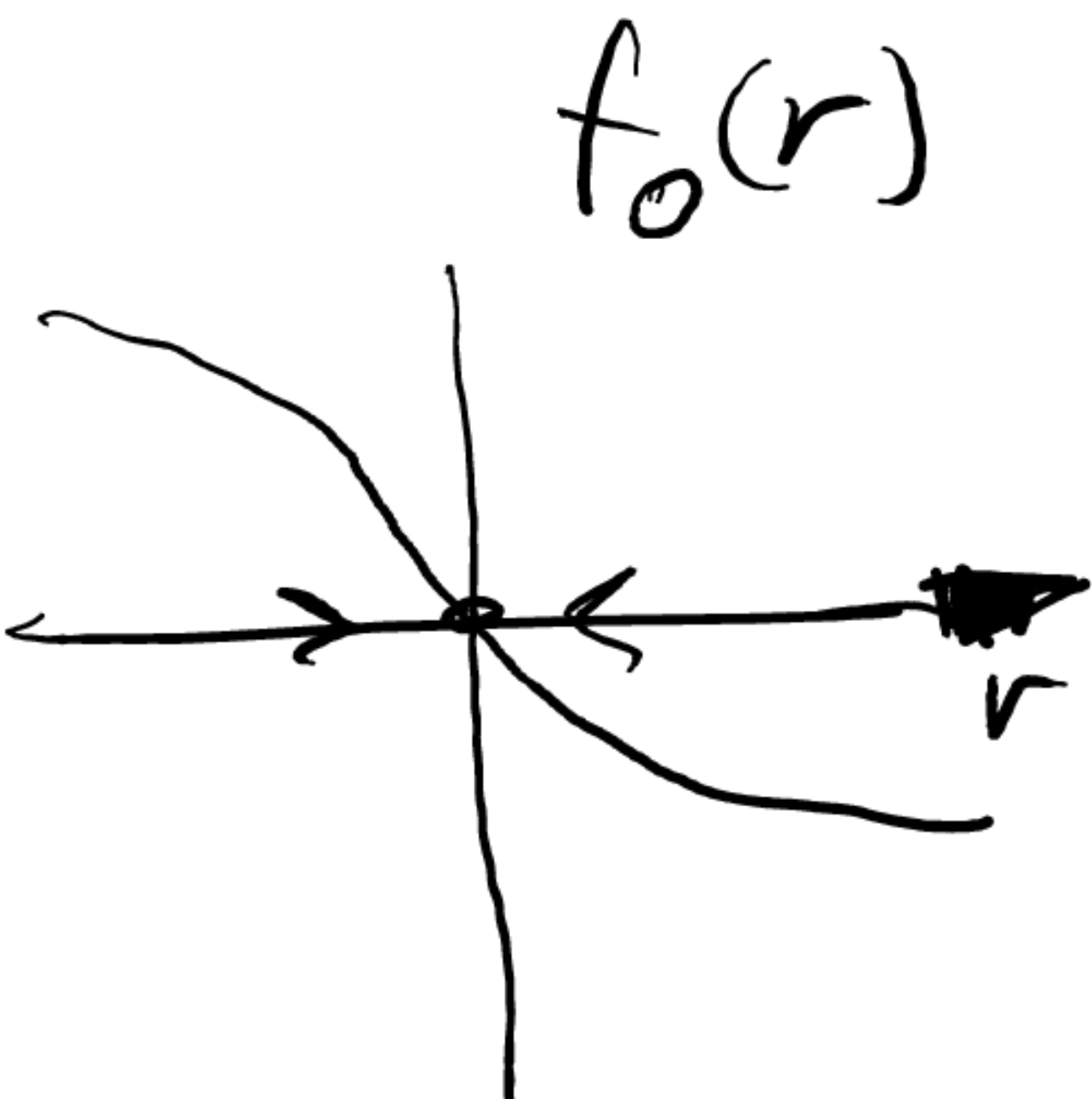
Recall (lecture 17)

$$\underline{X}^* = 0 \text{ A-stable} \iff \lim_{t \rightarrow \infty} r(t) = 0$$

for any (r_0, θ) with r_0 sufficiently small.

Stability of $\underline{X}^*$ determined by $f_a(r)$.

$f'_a(0) = a$ and for $a = 0$



$\bar{x}^* \& r^*$ is $\begin{cases} \text{a sink} & \text{if } a < 0 \\ \text{A-stable} & \text{if } a = 0 \\ \text{a source} & \text{if } a > 0 \end{cases}$

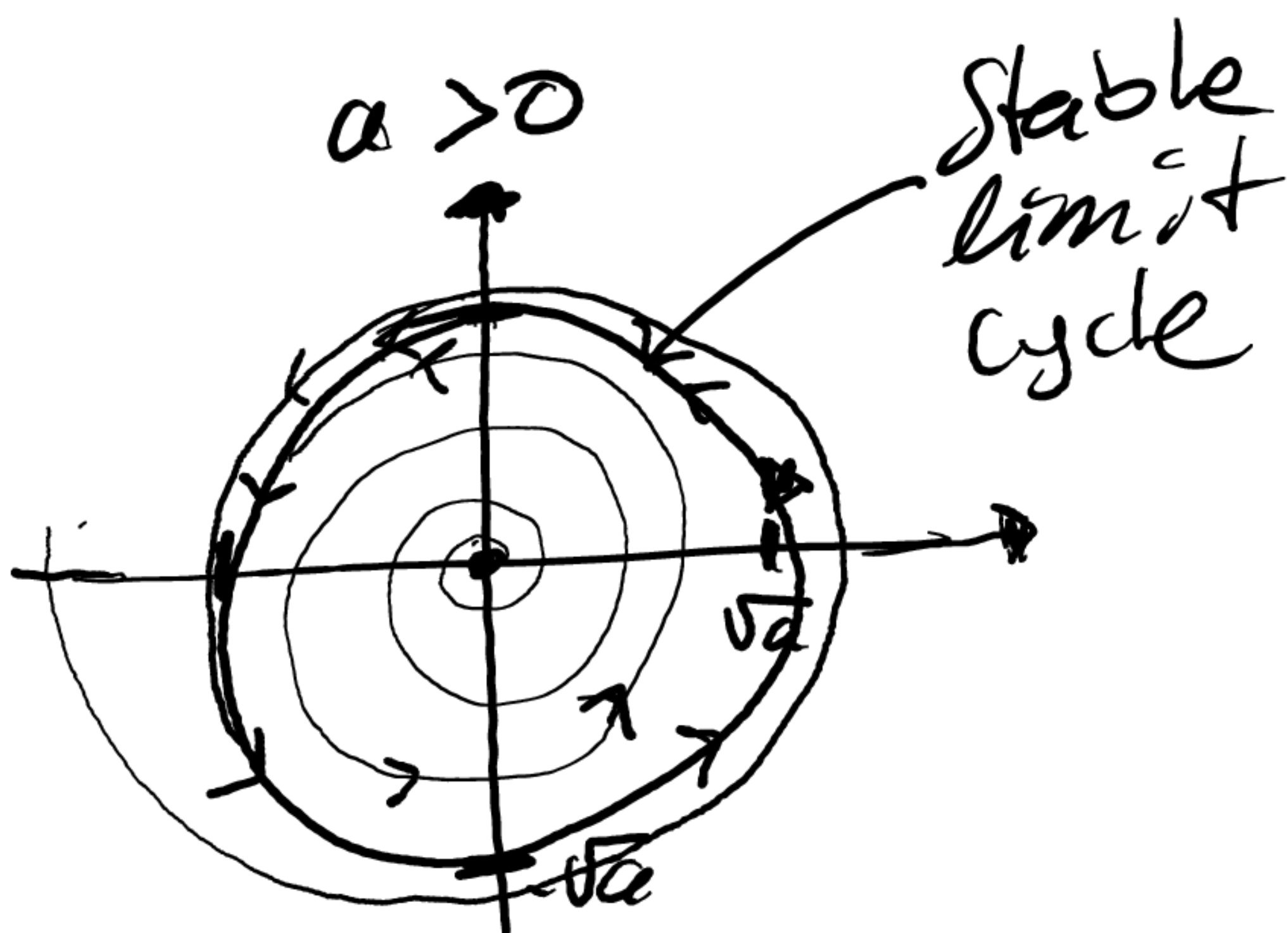
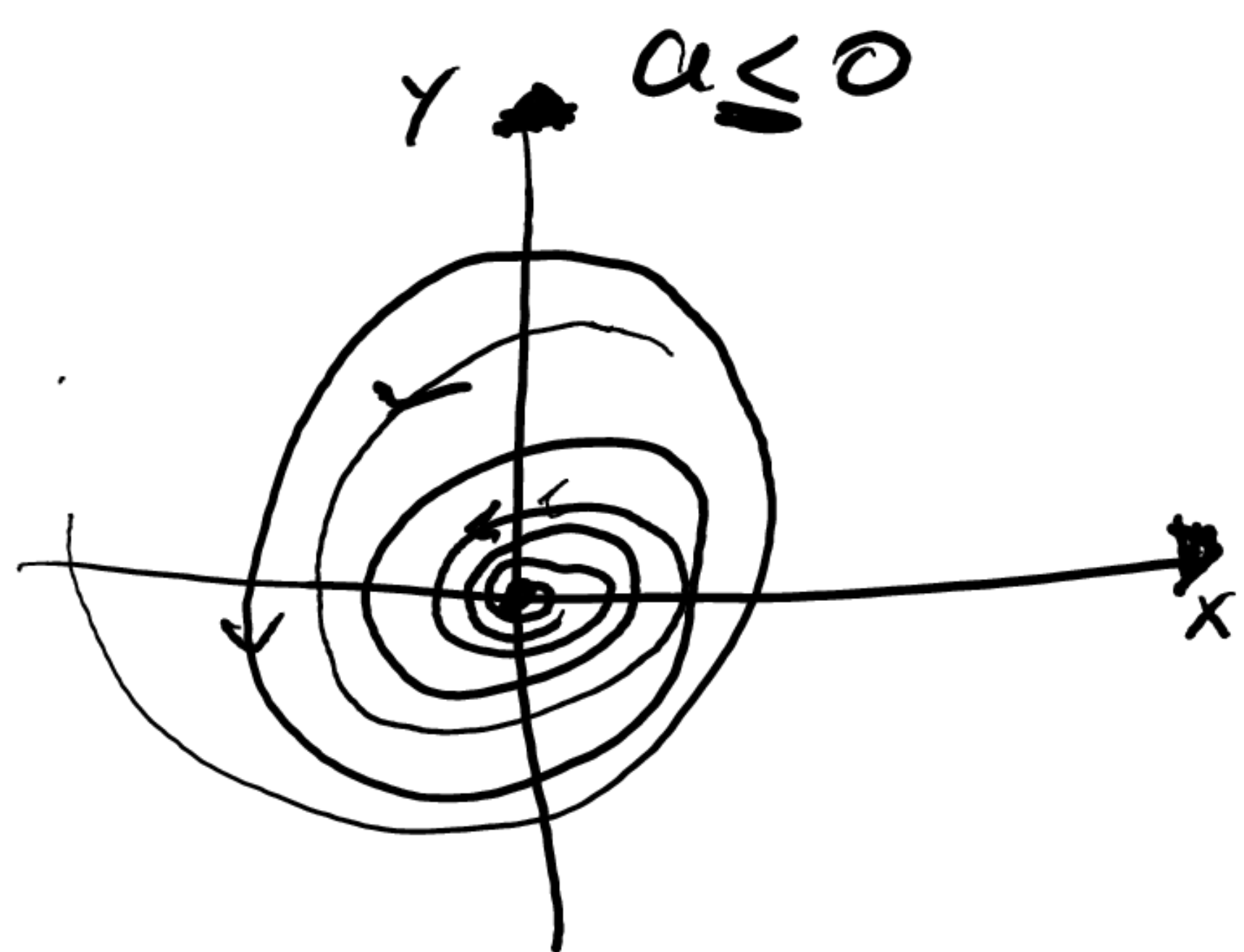
and it is a spiral, since $\theta' \neq 0 \forall t$.

Further, when $a > 0$,

$f_a(r) = ar - r^3$ has two roots:

$$r_0^* = 0, \quad r_1^* = \sqrt{a}$$

and $f'_a(r_1^*) < 0 \Rightarrow r_1^*$ is A-stable



("Stable" means nearby dynamics spiral toward it.)

Hopf	$a \leq 0$	$\bar{x}^* = 0$ stable
bifurcation	$a > 0$	$\bar{x}^* = 0$ unstable + stable limit cycle $r(t) = \sqrt{a}$ $\theta(t) = \theta_0 + t$

Application (related to ODE §12.4)

Circuit theory

$x(t)$ = current

$y(t)$ = voltage

$a \in \mathbb{R}$ is resistor parameter (e.g. temperature in resistor)

Circuit dies:

If $a \leq 0$, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \rightarrow 0$ as $t \rightarrow \infty$

Circuit oscillates:

If $a > 0$ & $\bar{x}_0 \neq 0$, then

$$x(t) = r(t) \cos(t + \theta_0)$$

$$y(t) = r(t) \sin(t + \theta_0)$$

with $r(t) \rightarrow \sqrt{a}$ as $t \rightarrow \infty$.

Lyapunov stability

Tool to verify stability of EP $\bar{x}^*$ of

$$\dot{x} = F(x)$$

(different from linearization)

Let $D \subset \mathbb{R}^n$ be open and contain $\bar{x}^*$.

For any $L \in C^1(D)$, note that

$$\frac{d}{dt} L(x(t)) = \underbrace{\nabla L(x(t))}_{\text{Gradient of } L} \cdot F(x(t))$$

For any point $\gamma \in D$, we define

$$\dot{L}(\gamma) := \nabla L(\gamma) \cdot F(\gamma) \quad (\dot{L} : D \rightarrow \mathbb{R})$$

Def 1 A function $L \in C^1(D)$ satisfying $L(x^*) = 0$ and

a) $L(x) > 0 \quad \forall x \in D \setminus \{x^*\}$

b) $\dot{L}(x) = \nabla L(x) F(x) \leq 0$

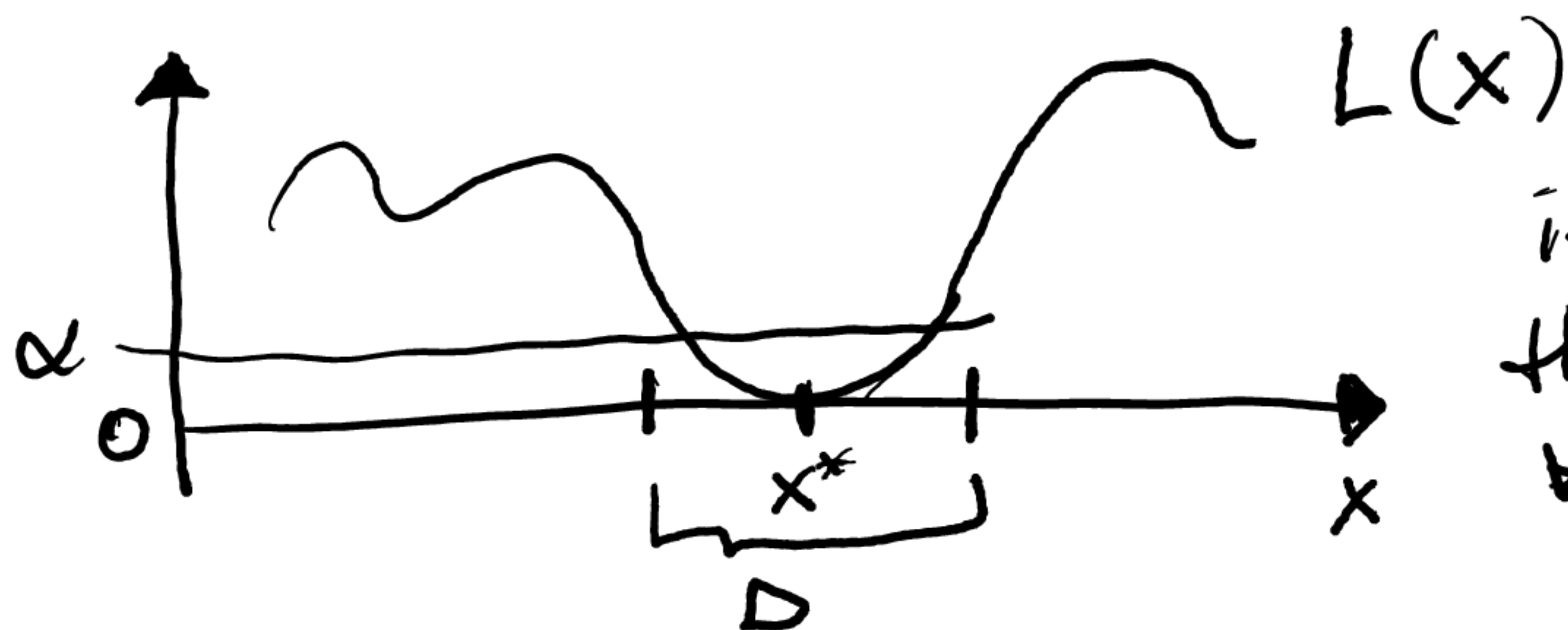
for all $x \in D \setminus \{x^*\}$ is called a Lyapunov function for x^* .

If also

c) $\dot{L}(x) < 0 \quad \forall x \in D \setminus \{x^*\}$,

then L is a strict Lyapunov function for x^* .

(Note Def 1 is associated to a given ODE $x' = F(x)$ with an EP x^* .)



if $x_0 \in D$
then $L(x(t)) \leq \alpha$
 $\forall t \Rightarrow x(t) \in D \quad \forall t$

Theorem 1 Let F be C^1 and consider $\dot{X} = F(X)$ with an EP X^* .

If L is a Lyapunov function for X^* ,
then X^* is stable.

And if L is strict Lyapunov for X^* ,
then X^* is A-stable.

Ex 1

$\dot{x} = -x^3$ has EP $x^* = 0$.

$L(x) = x^2$ is a strict Lyapunov fun
for 0 as $L(0) = 0$, $L(y) > 0 \forall y \neq 0$
and $\dot{L}(y) = L'(y)f(y) = -2y^4 < 0 \forall y \neq 0$.

Thm 1 $\Rightarrow$ 0 is A-stable.

(Can also see this from phase line ).

Linearization: $\tilde{x}' = -f'(0)\tilde{x} = 0$ does
not "work" as $f'(0)$ is not hyperbolic.)

Ex 2

$$\dot{X} = \begin{bmatrix} x_2 \\ -x_1^3 - x_2 \end{bmatrix} = F(X)$$

$$L(X) = \frac{x_1^4}{4} + \frac{x_2^2}{2} \quad \text{is Lyapunov fun}$$

$$\text{for } X^* = 0 \quad \text{as } L(0) = 0,$$

$$L(X) > 0 \quad \forall X \in \mathbb{R}^2 \setminus \{0\}.$$

$$\dot{L}(X) = \nabla L(X) \cdot F(X) = \begin{bmatrix} x_1^3 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ -x_1^3 - x_2 \end{bmatrix}$$

$$= -x_2^2 \leq 0 \quad \forall X \in \mathbb{R}^2$$

(near 0 would suffice).

$\Rightarrow 0$ is a stable EP.

Exercise Show that 0 is A-stable using another Lyapunov function.

Proof Thm 1



Let $d > 0$ be such that $\overline{B_d(\bar{x}^*)} \subset D$

and set $\alpha := \min_{\bar{y} \in \partial B_d(\bar{x}^*)} L(\bar{y}) \stackrel{\text{Def 1a)}}{>} 0$

Let $A := \{\bar{y} \in \overline{B_d(\bar{x}^*)} \mid L(\bar{y}) < \alpha\}$

(nonempty open, as L continuous.)

Claim For any $\bar{x}_0 \in A$, $\bar{x}(t) \in B_d(\bar{x}^*) \forall t \geq 0$

Pf. Suppose $\bar{x}(s) \in \partial B_d(\bar{x}^*)$ for some $s > 0$.

Then $L(\bar{x}(s)) \geq \alpha$, while $\bar{x}_0 \in A$
 $\Rightarrow L(\bar{x}_0) < \alpha$. Contradiction:

$$\begin{aligned} L(\bar{x}(s)) &= L(\bar{x}_0) + \int_0^s \frac{d}{dt} L(\bar{x}(t)) dt \\ &< \alpha + \int_0^s \underbrace{\dot{L}(\bar{x}(t))}_{\leq 0 \text{ Def 1b)}} dt \leq \alpha \quad \forall s \geq 0. \end{aligned}$$

By continuity of L and $L(\bar{x}^*) = 0$,

$\exists \bar{\delta} > 0$ s.t. $B_{\bar{\delta}}(\bar{x}^*) \subset A$,

(read: given $\alpha > 0$, $\exists \bar{\delta} > 0$ s.t.
 $L(\bar{x}) - L(\bar{x}^*) < \alpha \quad \forall \bar{x} \in B_{\bar{\delta}}(\bar{x}^*)$)

so

$\bar{x}_0 \in B_{\bar{\delta}}(\bar{x}^*) \Rightarrow \bar{x}(t; \bar{x}_0) \in B_{\bar{\delta}}(\bar{x}^*)$

$\forall t \geq 0$.

Next, let L strict Lyapunov and

$\bar{x}_0 \in B_{\bar{\delta}}(\bar{x}^*)$. Then $\bar{x}(t; \bar{x}_0) \in B_{\bar{\delta}}(\bar{x}_0) \forall t \geq 0$

(by above argument).

Then $L(\bar{x}_0) \stackrel{\text{Def. 1a)}}{\geq} 0$ and

$$\frac{d}{dt} L(\bar{x}(t)) = \dot{L}(\bar{x}(t)) \leq 0$$

so $L(\bar{x}(t))$ monotone decreasing
with $\inf_{t \geq 0} L(\bar{x}(t)) \geq 0$.

By MCT, there exists a limit

$$\lim_{t \rightarrow \infty} L(X(t)) =: \beta \geq 0.$$

Suppose $\beta > 0$. Then

$$\begin{aligned} X(t) &\in \{Y \in \overline{B}_\delta(X_0) \mid L(Y) \geq \beta\} \\ &= \underbrace{\overline{B}_\delta(X_0) \cap L^{-1}([\beta, \infty))}_{=: D_\beta \text{ compact.}} \quad \forall t \geq 0 \end{aligned}$$

Since $X^* \notin D_\beta$, Def 1c) implies

that $\min_{Y \in D_\beta} \dot{L}(Y) =: \gamma < 0.$

Leading to the contradiction

$$\begin{aligned} \beta &\leq L(X(s)) = L(X(0)) + \int_0^s \left(\frac{d}{dt} L(X(t)) \right) dt \\ &= L(X_0) + \int_0^s \underbrace{\dot{L}(X(t))}_{\in D_\beta} dt = L(X_0) - s\gamma \rightarrow -\infty \\ &\quad \text{as } s \rightarrow \infty \end{aligned}$$

(contradicting $\beta > 0$)

Hence $\beta = 0$ ✓

$$\lim_{t \rightarrow \infty} L(x(t)) = 0 \xRightarrow{\text{Def 1a}} x(t) \rightarrow x^*$$

for any $x_0 \in B_{\delta}(x^*)$.

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