

lecture 21

Exercise 8.1 a) Find all EP, describe behavior of linearized system for each EP.
 c) Does linearized system accurately describe behavior of nonlinear system locally?

$$(i) \begin{cases} x' = \sin x \\ y' = \cos y \end{cases}$$

$$EP: \bar{x}_{ij} = (x_i, y_j) = (i\pi, \frac{\pi}{2} + j\pi) \quad i, j \in \mathbb{Z}$$

$$F(\bar{x}) = \begin{bmatrix} \cos x & 0 \\ 0 & -\sin y \end{bmatrix}$$

$$DF(\bar{x}_{ij}) = \begin{bmatrix} (-1)^i & 0 \\ 0 & (-1)^j \end{bmatrix}$$

Notation $\tilde{x}_{ij}(t) = \begin{bmatrix} \tilde{x}_{ij} \\ \tilde{y}_{ij} \end{bmatrix}$ and linearization

$$\tilde{x}_{ij}' = DF(\bar{x}_{ij})(\tilde{x}_{ij} - \bar{x}_{ij}) = \begin{bmatrix} (-1)^i (\tilde{x}_{ij} - x_i) \\ (-1)^j (\tilde{y}_{ij} - y_j) \end{bmatrix}$$

For $\tilde{x}_{ij}(t)$, $\bar{x}_{ij}$ is a $\begin{cases} \text{sink} & \text{when } i, j \text{ odd} \\ \text{source} & \text{when } i, j \text{ even} \\ \text{saddle} & \text{otherwise} \end{cases}$

c) $DF(\bar{x}_{ij})$ is hyperbolic for all i, j ,
so $\tilde{x}_{ij}$ approximates $\bar{x}$ well near $\bar{x}_{ij}$.

$$\text{ii) } \begin{aligned} x' &= x(x^2 + y^2) \\ y' &= y(x^2 + y^2) \end{aligned}$$

Only EP $\bar{x}^* = 0$. $DF(\bar{x}) = \begin{bmatrix} 3x^2 + y^2 & 2xy \\ 2xy & 3y^2 + x^2 \end{bmatrix}$

$$\begin{aligned} \tilde{x}' &= DF(0) \tilde{x} \\ &= 0 \end{aligned}$$

Linearized system is constant in time:

$$\tilde{x}(t) = \tilde{x}(0)$$

c) Consider rather $z(t) = \frac{|\bar{x}|^2}{2} = \frac{x^2 + y^2}{2}$

$$\Rightarrow z'(t) = xx' + yy' = |\bar{x}|^4 = z^2$$

So for any $z(0) \neq 0$, $z(t)$ blows up in finite time. Hence $\bar{x}^* = 0$ is unstable.

For $\tilde{x}$, 0 is stable, so $\tilde{x} \neq \bar{x}$ near 0.

$$(ii) \quad x' = x + y^2$$

$$a) \quad y' = 2y$$

$$EP: 2y = 0 \Leftrightarrow y^* = 0. \text{ So then } x^* + (y^*)^2 = 0 \\ \Leftrightarrow x^* = 0.$$

Only EP: 0

$$DF(0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \tilde{x} \approx DF(0) \tilde{x}$$

0 is source for $\tilde{x}$ and

c) $\tilde{x} \approx \tilde{x}$ near 0 as $DF(0)$ is hyperbolic.

2. Find a global change of coordinates that linearizes

$$x' = x + y^2$$

$$y' = -y$$

$$z' = -z + y^2$$

Hint: mimic P. 161:
Solve exactly & find
stable curve.

sol First find exact solution (to get an idea of what might work)

$$Y(t) = Y_0 e^{-t} \quad \text{integrating factors}$$

$$X'(t) = X + Y_0^2 e^{-2t} \quad \left| \begin{array}{l} e^{-t} \\ e^t \end{array} \right.$$

$$Z'(t) = -Z + Y_0^2 e^{-2t}$$

$$X(t) = X_0 e^t + \frac{Y_0^2}{3} (e^t - e^{-2t})$$

$$= \left(X_0 + \frac{Y_0^2}{3} \right) e^t - \frac{Y_0^2}{3} e^{-2t}$$

$$Z(t) = Z_0 e^{-t} + Y_0^2 (e^{-t} - e^{-2t})$$

$$= (Z_0 + Y_0^2) e^{-t} - \frac{Y_0^2}{3} e^{-2t}$$

$$\text{stable curve : } \left\{ (x, y, z) \mid \begin{array}{l} x + y^2/3 = 0 \\ z + y^2 = 0 \end{array} \right\}$$

Change of coordinates

$$u = x + y^2/3, \quad v = y, \quad w = z + y^2$$

$$\Rightarrow u' = x' + \frac{2}{3} y y' = x + y^2/3 = u$$

$$v' = -y = -v$$

$$w' = z' + 2y y' = -(z + y^2) = -w.$$

8.3 Consider $x' = f_a(x) = g(x, a)$

$f_a(x_0) = 0$ and $f'_a(x_0) \neq 0$. Prove that

$x' = f_{a+\epsilon}(x)$ has an EP $x_0(\epsilon)$ for ϵ near 0.

Sol $g(x_0, a) = 0$ & $g_x(x_0, a) \neq 0$ yields

by IFT that $\exists \epsilon \mapsto x_0(\epsilon)$ s.t.

$g(x_0(\epsilon), a + \epsilon) = 0$ for ϵ near 0

with $x_0(0) = 0$. And g smooth

$\Rightarrow x_0(\epsilon)$ smooth.

8.5 Consider $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x^2 + y \\ x - y + a \end{bmatrix} = F_a(X), a \in \mathbb{R}$

a) Find all EP, compute linearized ODE for each one.

b) Describe behavior of linearized system near each EP

c) Describe any bifurcations that occur.

$$\underline{\text{sol}} \quad \left. \begin{array}{l} a) - b) \quad x^2 + y = 0 \\ x - y + a = 0 \end{array} \right\} \Rightarrow \begin{array}{l} x^2 + x + a = 0 \\ y = -x^2 \end{array}$$

$$x_{\pm} = \frac{-1 \pm \sqrt{1-4a}}{2}$$

$$EP: \underline{x}_{\pm}^* = (x_{\pm}, -(x_{\pm})^2) \quad a < \frac{1}{4}$$

$$\underline{x}^* = \left(-\frac{1}{2}, -\frac{1}{4}\right) \quad a = \frac{1}{4}$$

No EP for $a > \frac{1}{4}$.

so $a = \frac{1}{4}$ is a bifurcation point.

$$\text{Linearizations: } DF(\underline{x}) = \begin{bmatrix} 2x & 1 \\ 1 & -1 \end{bmatrix}$$

$$\tilde{\underline{x}}_+^1 = \begin{bmatrix} -1 + \sqrt{1-4a} & 1 \\ 1 & -1 \end{bmatrix} (\tilde{\underline{x}} - \underline{x}_+^*)$$

$$\det(DF(\underline{x}_+^*)) = -\sqrt{1-4a} \Rightarrow \lambda_1, \lambda_2 < 0$$

$\underline{x}_+^*$ is saddle for $\tilde{\underline{x}}_+$

$$\tilde{x}' = \begin{bmatrix} -1 - \sqrt{1-4a} & 1 \\ 1 & -1 \end{bmatrix} (\tilde{x} - x^*)$$

$$\lambda_1 \lambda_2 = \det(DF(x^*)) = \sqrt{1-4a} > 0$$

And $\lambda_1 + \lambda_2 = \text{Tr}(DF(x^*)) < 0 \Rightarrow \lambda_1, \lambda_2 < 0$

$\Rightarrow x^*$ is a sink for $\tilde{x}$.

Lastly, for $x^* = (-\frac{1}{2}, -\frac{1}{4})$,

$$\tilde{x}' = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} (\tilde{x} - x^*)$$

EV: $\lambda_1 \lambda_2 = \det(DF(x^*)) = -1 < 0$

so x^* is a saddle for $\tilde{x}$.

c) Bifurcation happens at $a = \frac{1}{4}$

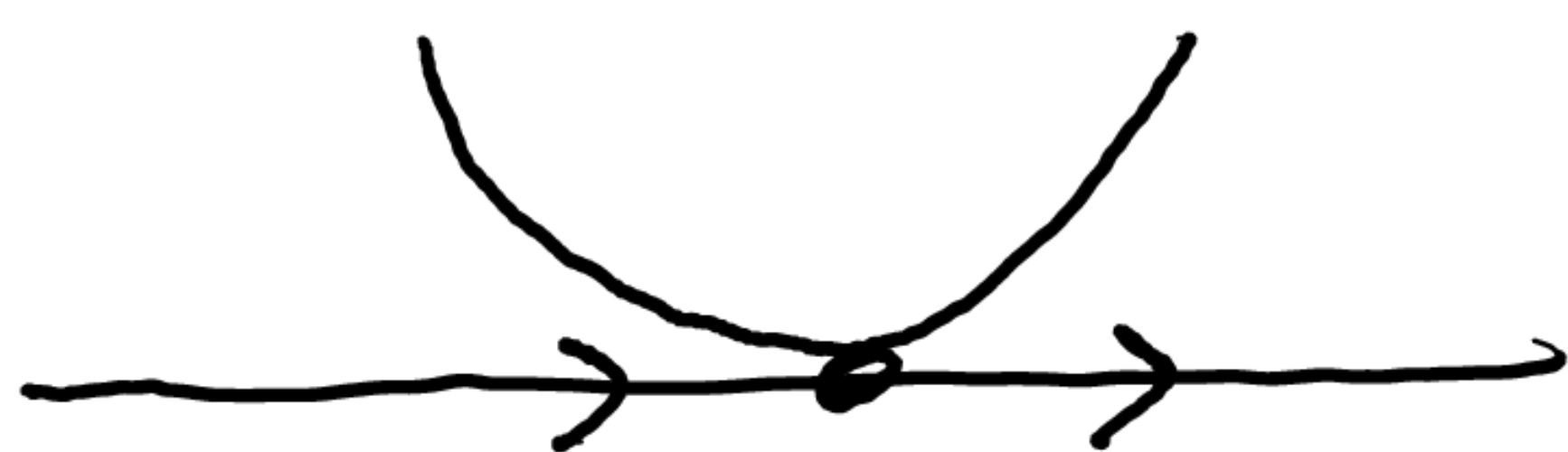
saddle-node for $a < \frac{1}{4}$

no EP for $a > \frac{1}{4}$; a saddle-node bifurcation.

8.6 Give an example of a family of ODE $x' = f_a(x)$ $a \in \mathbb{R}$ for which there are no EP for $a < 0$, one for $a = 0$ and four for $a > 0$.

Sol $x' = (x^2 - a)(x^2 - 4a) =: f_a(x)$

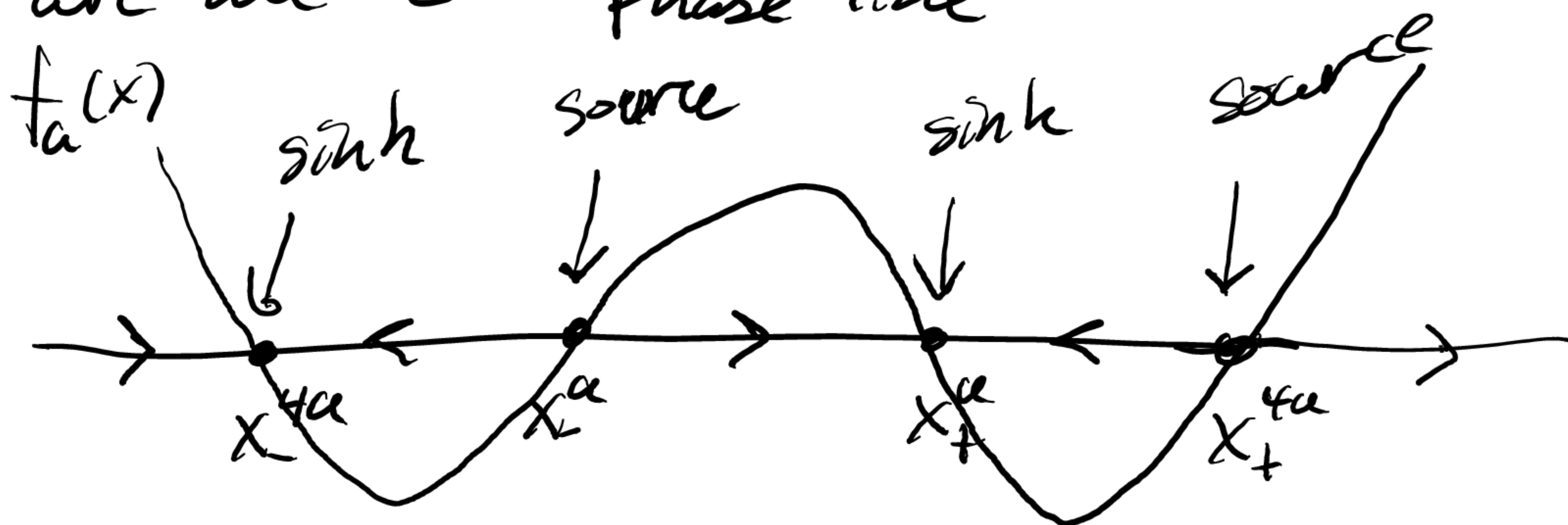
EP: $x_0 = 0$ for $a = 0$ $f_0(x) = x^4$

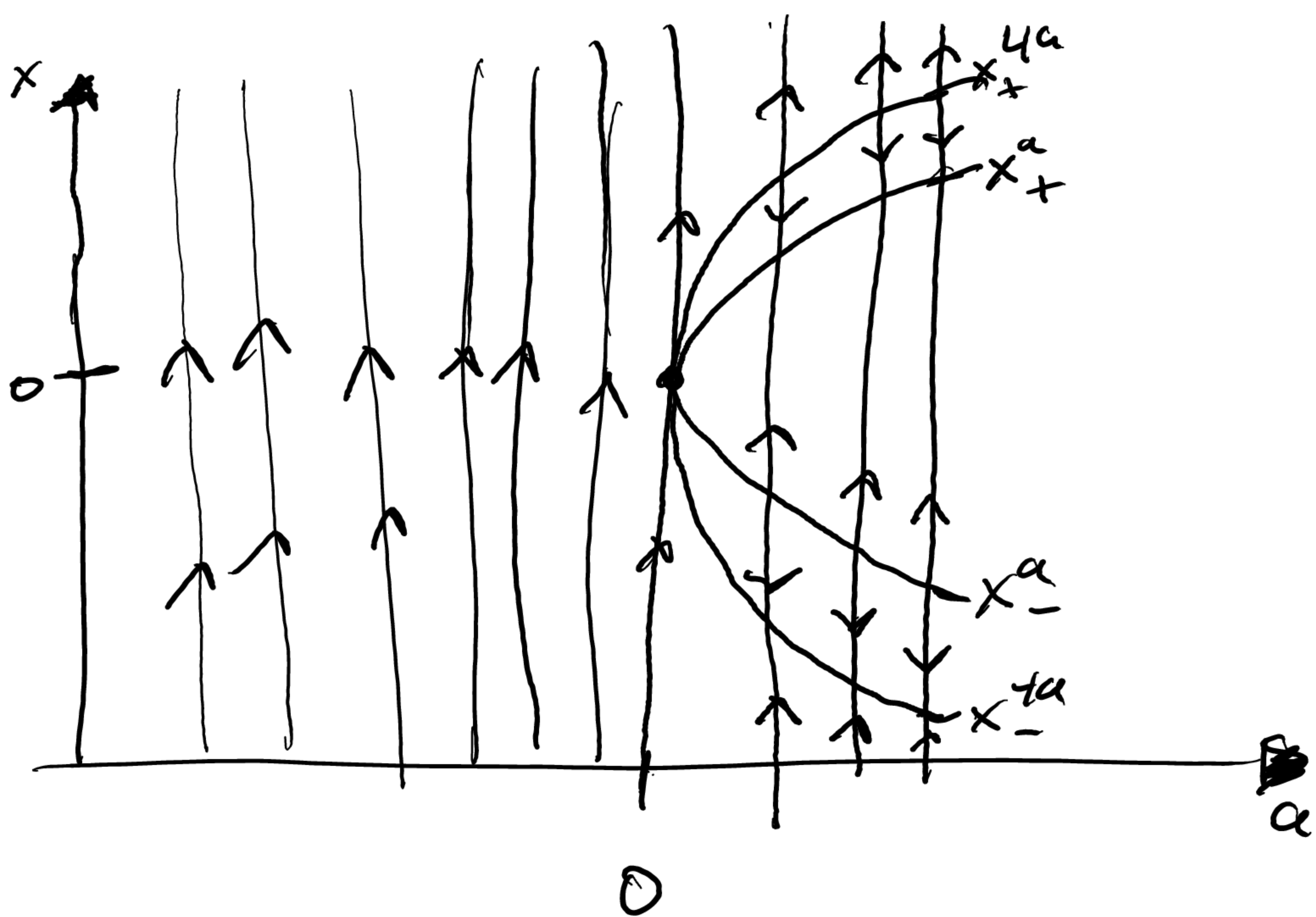


x_0 is unstable

$a > 0$: $x_{\pm}^a = \pm\sqrt{a}$, $x_{\pm}^{4a} = \pm 2\sqrt{a}$

are all EP. Phase line





8.7 Discuss the global behavior of solutions of

$$r' = r - r^3 = f_1(r)$$

$$\theta' = \sin^2(\theta) + a = f_2(\theta)$$

at the bifurcation value $a = -1$

EP: In Cartesian coordinates $x = r \cos \theta$
 $y = r \sin \theta$,

we see that $\bar{x}^* = 0$ is an EP
(lecture 17).

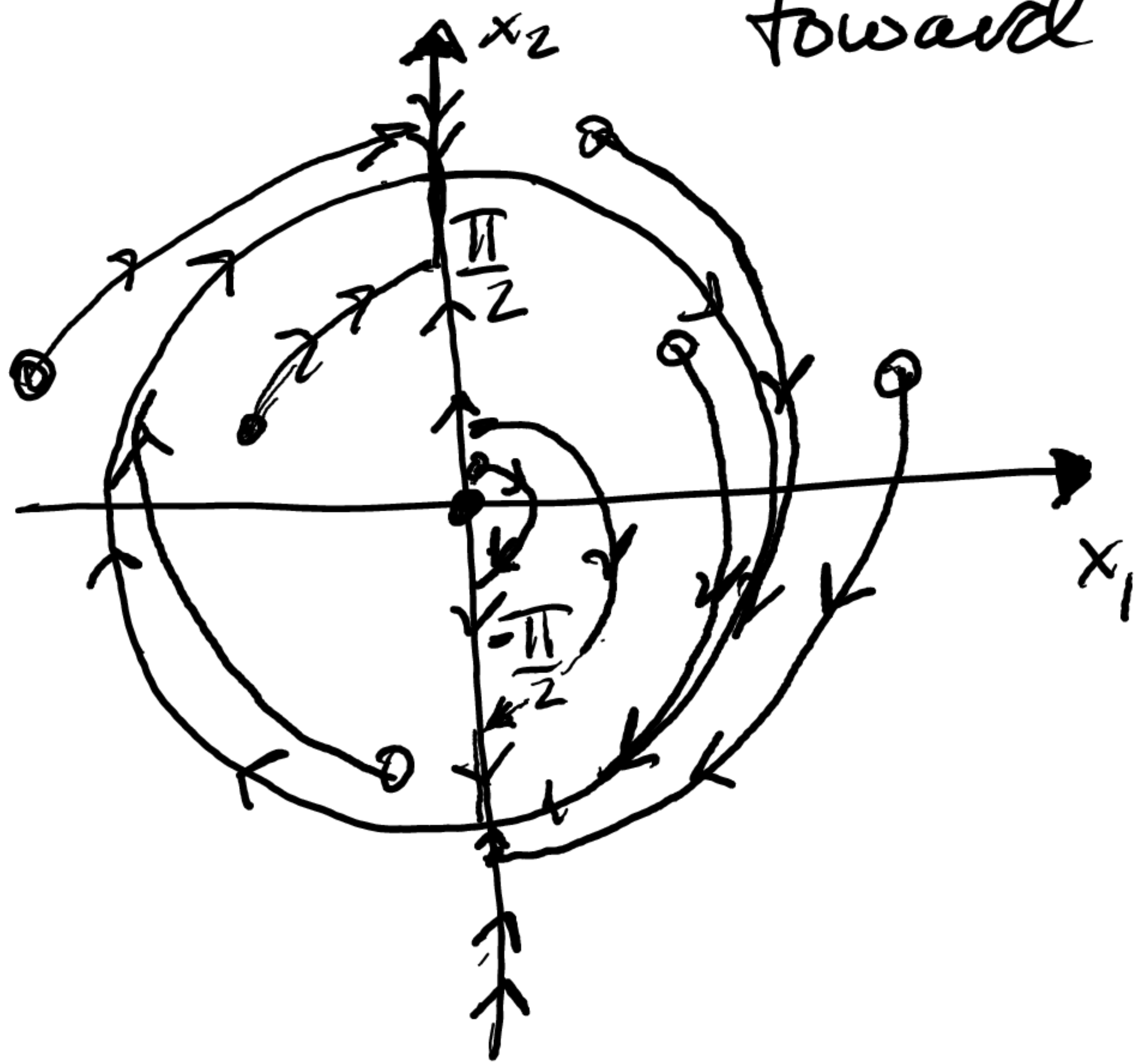
It is a source since $r - r^3 > 0$
for $r \neq 0$ near 0.

And $(r_+, \theta_+) = (1, \frac{\pi}{2})$, $(r_-, \theta_-) = (1, -\frac{\pi}{2})$
are EP. $f'_1(1) = -1 \Rightarrow$ dynamics
is stable in the r -direction.

In the θ direction, rotation happens in the clockwise direction until $\theta = \theta_{\pm}$

$(r, \theta_{\pm})$ for $r \in [0, \infty)$ is the y -axis.

where dynamics for $r \neq 0$ is pushed toward $(1, \pm \frac{\pi}{2})$

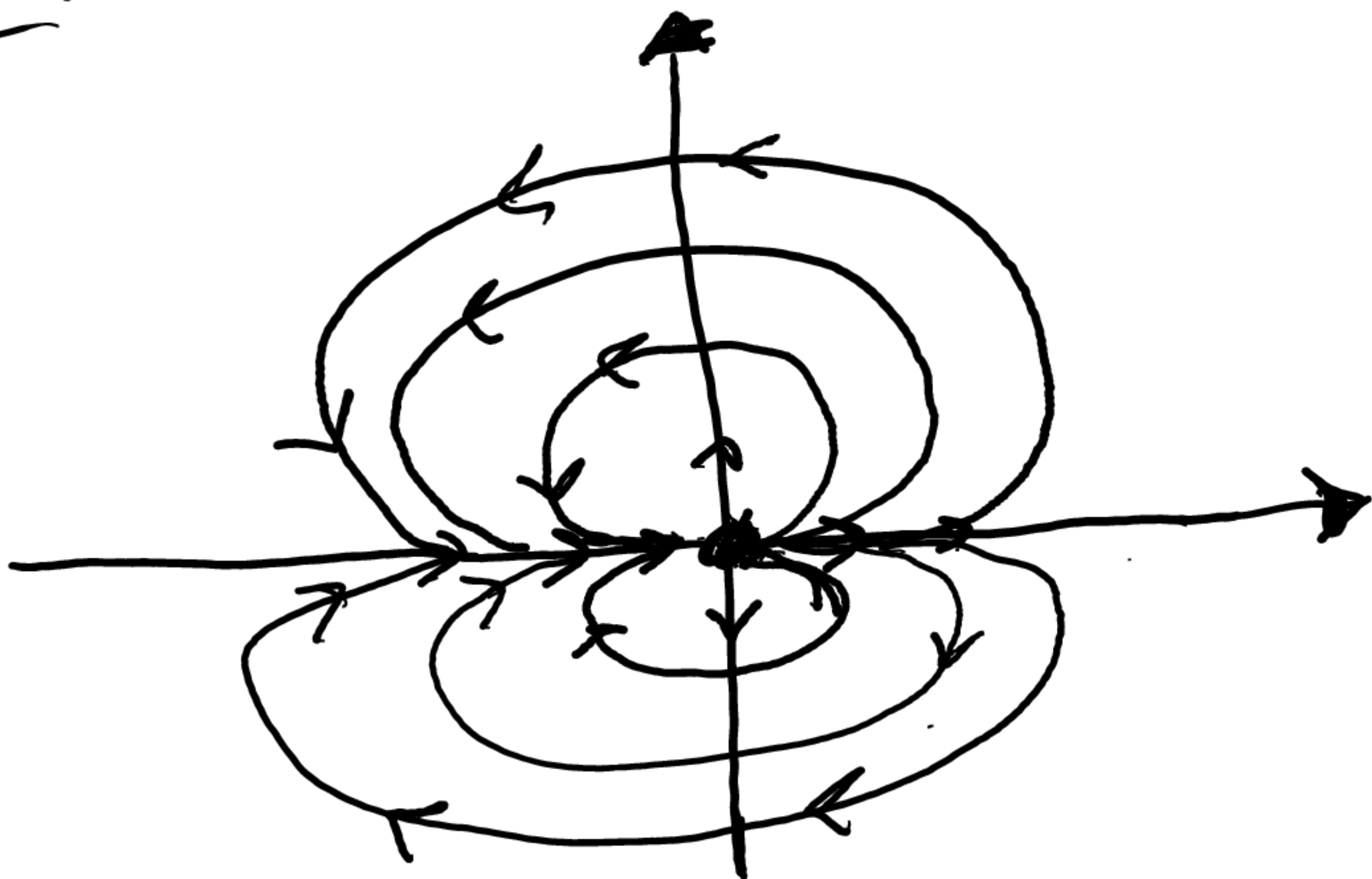


8.12 By definition X^* is A-stable if X^* is stable and $X(t; X_0) \rightarrow X^*$ for all X_0 near X^* .

Illustrate by a phase portrait that if we instead defined X^* is A-stable if $X(t; X_0) \rightarrow X^*$ for all X_0 near X^* , then

$\bar{x}^*$ -stable would not imply $\bar{x}^*$ stable.

Sol Let $\bar{x}^* = 0$ be EP and



All paths converge first to negative part of x -axis and then toward $\bar{x}^* = 0$.
But for values $\bar{x}_0$ arbitrarily close to $\bar{x}^*$
with $x_1 > 0$, $\sup_{t \in \mathbb{R}} |\bar{x}(t, \bar{x}_0)| > 1$.

(See magnetic dipole field)