

Lecture 22

Recall For $\dot{X} = F(X)$ on $\mathbb{R}^n$ with
EP X^* , L is a Lyapunov fun for X^*
if $\exists$ open $D \subset \mathbb{R}^n$ containing X^* s.t.
 $L \in C^1(D, \mathbb{R})$ satisfies $L(X^*) = 0$

a) $L(X) > 0$

b) $\dot{L}(X) = DL(X) \cdot F(X) \leq 0 \quad \forall X \in D - \{X^*\}$

c) If $\dot{L} < 0$ on b) then L is strict
Lyapunov.

Thm 1 If L is Lyapunov for X^* ,
then X^* is stable.

If L also is strict Lyapunov, then
 X^* is A-stable.

Remark A-stability holds under more
relaxed constraints than c)
(if $\dot{X} \neq 0$ when $L(X) = 0$ and $X \neq 0$)

(see Lasalle's invariance principle p. 199)

When c) holds, we can find $\delta > 0$
s.t. $\underline{x}(t; \underline{x}_0) \rightarrow \underline{x}^* \quad \forall \underline{x}_0 \in B_\delta(\underline{x}^*)$. (1)

If $\underline{x}^*$ is A-stable, then
 $A_{\underline{x}^*} := \{ \underline{x}_0 \in \mathbb{R}^n \mid \lim_{t \rightarrow \infty} \phi_t(\underline{x}_0) = \underline{x}^* \}$
is called $\underline{x}^*$'s basin of attraction

Hence by (1), $B_\delta(\underline{x}^*) \subset A_{\underline{x}^*}$
Lyapunov theory can be used to
approximate $A_{\underline{x}^*}$ from below.

Ex 1

$$\underline{x}' = \begin{bmatrix} -x_1 + x_1 x_2^2 \\ -x_2 + x_2 x_1^2 \end{bmatrix} = F(\underline{x}) \quad \text{EP } \underline{x}^* = 0$$

is A-stable by linearization: $DF(0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Estimate of A_0 from below:

Consider $L(x) = |x|^2$

$$\begin{aligned} \Rightarrow \dot{L}(x) &= \nabla L \cdot F = 2x \cdot \begin{bmatrix} -x_1 + x_1 x_2^2 \\ -x_2 + x_2 x_1^2 \end{bmatrix} \\ &= -2|x|^2 + 2x_1^2 x_2^2 \\ &\leq -2|x|^2 + (x_1^2 + x_2^2)^2 \\ &= |x|^2(|x|^2 - 2) \\ &< 0 \quad x \in D = B_{\frac{1}{\sqrt{2}}}(0) \setminus \{0\} \end{aligned} \quad (2)$$

$$L(0) = 0, \text{ \& } L(x) > 0 \text{ for } x \neq 0.$$

$\Rightarrow L$ strict Lyapunov for 0.

Generic basin estimation:

$$\text{Let } \alpha = \min_{\partial D} L(x). \text{ (Here: } \alpha = 2 \text{.)}$$

$$\text{Find } \delta > 0 \text{ s.t. } B_\delta(0) \subset L^{-1}([0, \alpha)).$$

Here $\delta = \sqrt{2}$. Then for $x_0 \in B_\delta(0)$,

$x(t) \notin \partial D$ for any time, as

$$L(x_0) < \min_{\partial D} L \quad \& \quad \dot{L}(x) < 0 \text{ in } D.$$

Further $x(t; x_0) \rightarrow x^*$, so $B_\delta(0) \subset A_0$.

Observe (simplification for this example):

Considering $z(t) = |x(t)|^2$,

$$(2) \Rightarrow z' = 2z(2z - 1)$$

So for $z > 1$, i.e. $|x|^2 > 2$,

$$\lim_{t \rightarrow \infty} \frac{|x(t)|}{2} = \lim_{t \rightarrow \infty} z(t) = \infty.$$

Q: How to find a suitable $L(x)$?

(possible for a class of problems
we consider next.)

Gradient systems

When $V: \mathbb{R}^n \rightarrow \mathbb{R}$ is smooth, the ODE

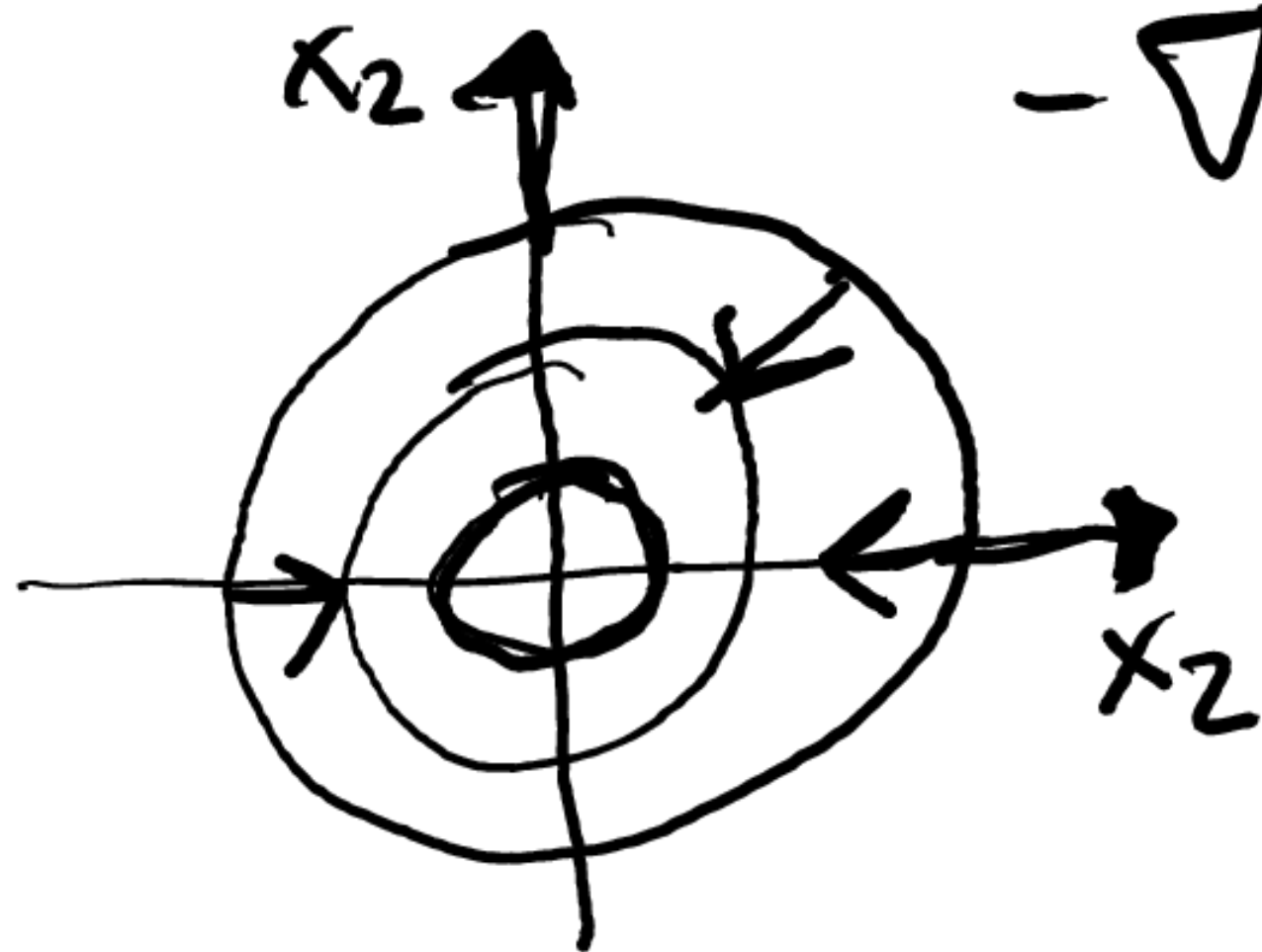
$\mathbf{x}' = -\nabla V(\mathbf{x})$ is called a gradient system/flow.

Properties

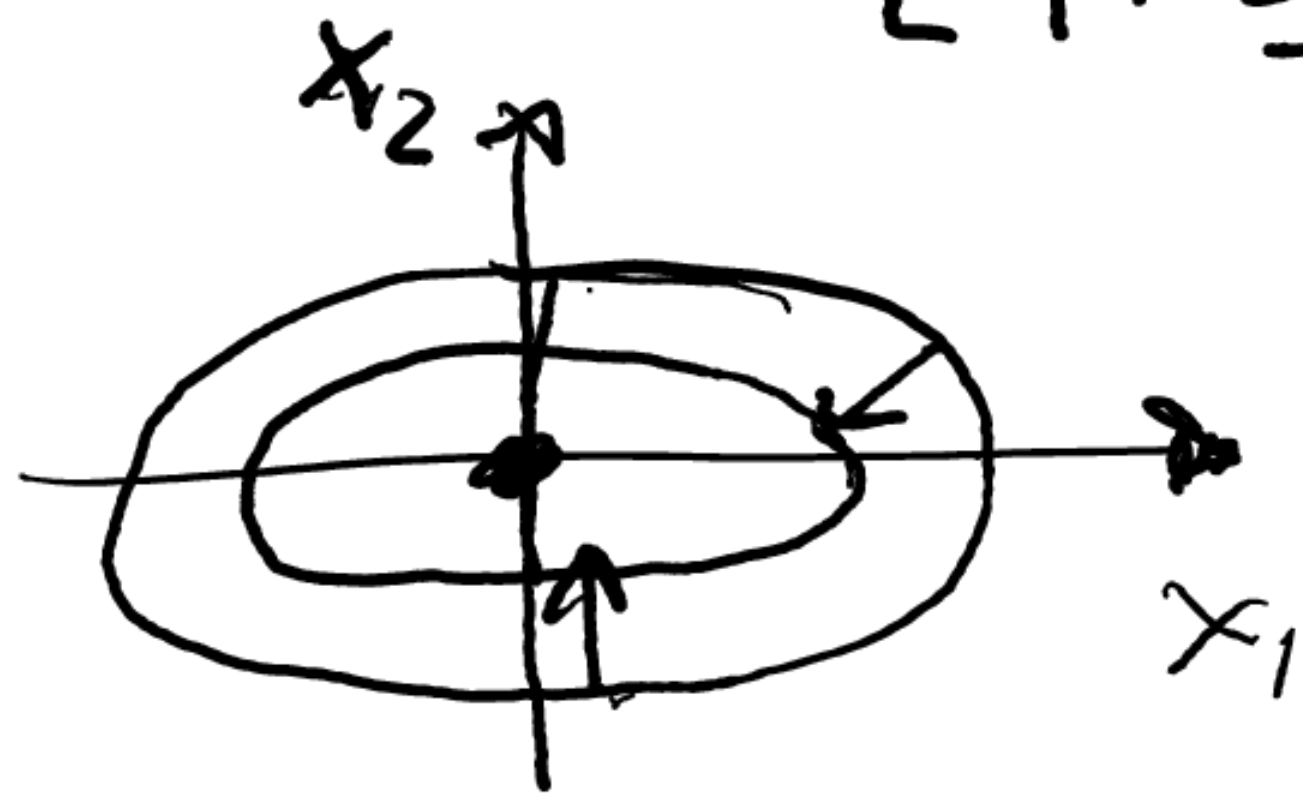
- $-\nabla V(\mathbf{x})$ = direction of steepest descent at point $\mathbf{x}$.
- $\mathbf{x}^*$ is an EP of $-\nabla V(\mathbf{x})$ iff it is a critical point (CP) of V (= local min/max or saddle point).

Ex $V(\mathbf{x}) = (x_1^2 + x_2^2)/2$

$$-\nabla V(\mathbf{x}) = -\mathbf{x}$$



$$V(\mathbf{x}) = x_1^2 + 2x_2^2$$
$$-\nabla V(\mathbf{x}) = -\begin{bmatrix} 2x_1 \\ 4x_2 \end{bmatrix}$$



(For $\mathbf{x} \in V^{-1}(c)$, $\nabla V(\mathbf{x}) \perp$ level set at $\mathbf{x}$.)

If $\mathbf{x}(t)$ solves ODE, then

$$\begin{aligned}\frac{d}{dt} V(\mathbf{x}(t)) &= \nabla V(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \\ &= -|\nabla V(\mathbf{x}(t))|^2 \leq 0\end{aligned}$$

Notation $\dot{V}(\mathbf{y}) := -|\nabla V(\mathbf{y})|^2$.

Observe $\dot{V}(\mathbf{y}) = \begin{cases} 0 & \Leftrightarrow \mathbf{y} \text{ is a CP} \\ < 0 & \text{otherwise.} \end{cases} \quad (3)$

So $V(\mathbf{x}(t))$ is monotonically decreasing.

(Consequence: $\mathbf{x}^*$ local min of V
 $\Rightarrow \mathbf{x}^*$ is EP of dynamics.)

Theorem 2 If $\mathbf{x}^*$ is strict local min of $V(\mathbf{x})$, then $\bar{V}(\mathbf{x}) := V(\mathbf{x}) - V(\mathbf{x}^*)$ is a strict Lyapunov function and $\mathbf{x}^*$ is A-stable EP of $\mathbf{x}' = -\nabla V(\mathbf{x})$

Proof $\bar{V}(\mathbf{x}^*) = 0$, $\bar{V}(\mathbf{x}) > 0$ and $\dot{\bar{V}}(\mathbf{x}) < 0$ for $\mathbf{x} \neq \mathbf{x}^*$ near $\mathbf{x}^*$. $\square$

Ex 2 Consider

$$V\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2(x-1)^2 + y^2 \geq 0 \text{ with}$$

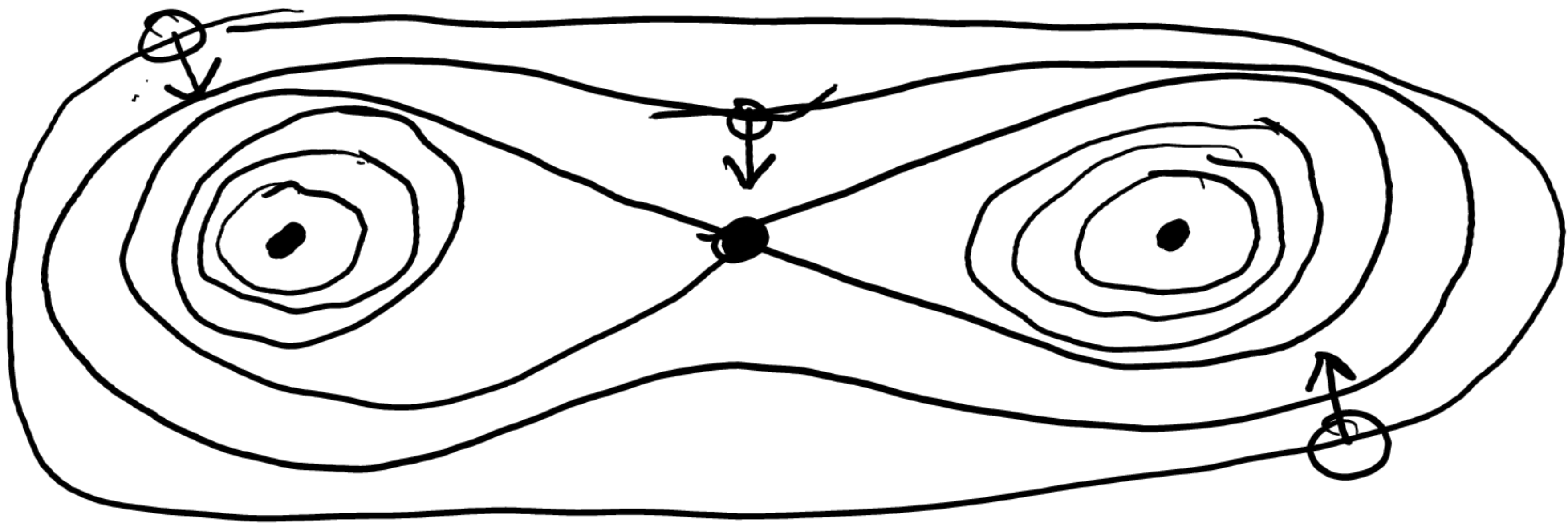
$$\nabla V(\mathbf{x}) = \begin{bmatrix} 2x(x-1)(2x-1) \\ 2y \end{bmatrix}$$

$$\text{CP: } (0,0), \left(\frac{1}{2}, 0\right), (1,0)$$

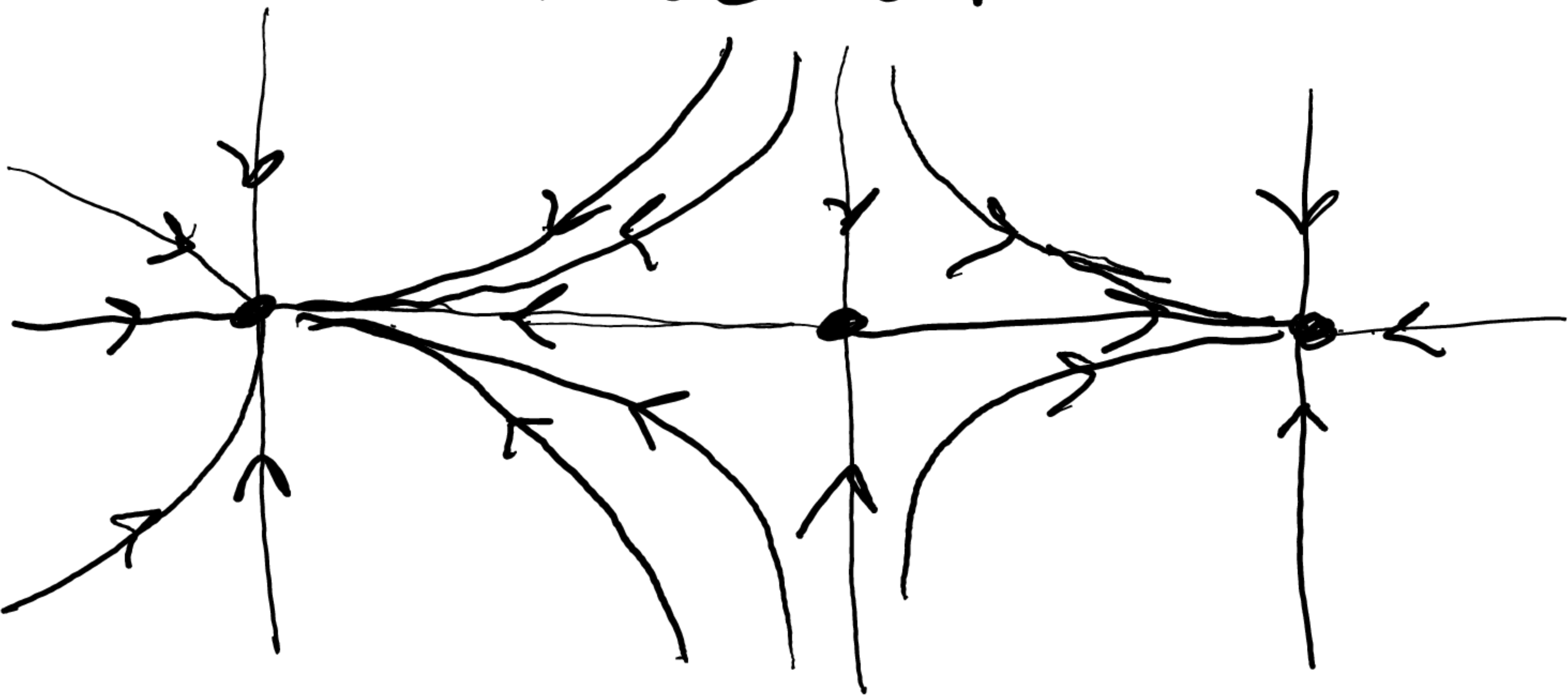
$(0,0)$ & $(1,0)$ are strict local minima,

Thm 2 $\Rightarrow$ both are A-stable EP of $\mathbf{x}' = -\nabla V(\mathbf{x})$.

Level sets



Phase portrait



Application related to Machine Learning:
 Find extreme values of $V(\mathbf{x})$ through
 gradient flow long-time simulation.

Def (Accumulation points)

Given ODE $\mathbf{x}' = F(\mathbf{x})$ and $\mathbf{x}_0 \in \mathbb{R}^n$, let

$$\omega(\mathbf{x}_0) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \exists \{t_n\} \subset \mathbb{R} \text{ with } t_n \rightarrow \infty \right. \\ \left. \text{s.t. } \lim_{n \rightarrow \infty} \mathbf{x}(t_n; \mathbf{x}_0) = \mathbf{x} \right\}$$

$$\alpha(\mathbf{x}_0) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \exists \{t_n\} \subset \mathbb{R} \text{ with } t_n \rightarrow -\infty \right. \\ \left. \text{s.t. } \lim_{n \rightarrow \infty} \mathbf{x}(t_n; \mathbf{x}_0) = \mathbf{x} \right\}$$

Ex 3 $x' = -x + \sin t + \cos t$

Solution $x(t; x_0) = x_0 e^{-t} + \sin t$

$$\omega(x_0) = [-1, 1] \quad (\text{set } t_n = \sin^{-1}(c) + n \cdot 2\pi \text{ to} \\ \text{get } \phi_{t_n}(x_0) \rightarrow c \in [-1, 1])$$

$$\alpha(x_0) = \begin{cases} [-1, 1] & \text{if } x_0 = 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

(as then $|x(t; x_0)| \rightarrow \infty$ for any x_0)

Lemma | For any autonomous ODE $x' = F(x)$, the set $\omega(x_0)$ is positively invariant, meaning $\gamma \in \omega(x_0) \Rightarrow \phi_t(\gamma) \in \omega(x_0) \quad \forall t \geq 0$,
(or $\phi_t(\omega(x_0)) \subset \omega(x_0) \quad \forall t \geq 0$)
Similarly, $\alpha(x_0)$ is negatively invariant:
 $\phi_t(\alpha(x_0)) \subset \alpha(x_0) \quad \forall t \leq 0$.

Proof When $\omega(x_0)$ is non-empty, then $\phi(t, x_0)$ must exist for all $t \geq 0$, and for any fixed $t \geq 0$, $\nabla V(x)$ smooth $\Rightarrow \phi(t, x)$ continuous in x . Let $\gamma \in \omega(x_0)$.

Then $\lim_{n \rightarrow \infty} \phi(t_n, x_0) = \gamma$ for some $\{t_n\} \Rightarrow$

$$\phi(t, \gamma) = \lim_{n \rightarrow \infty} \phi(t, \phi(t_n, x_0)) \quad (4)$$

Flow Property $\lim_{n \rightarrow \infty} \phi(\underbrace{t+t_n}_{s=s_n}, x_0) =$

So $\gamma \in \omega(x_0) \Rightarrow \phi(t, \gamma) \in \omega(x_0) \quad \square$

Notation sets of all forward/backward accumulation points:

$$\omega\text{-limit set } \mathcal{L}_\omega := \bigcup_{x_0 \in \mathbb{R}^n} \omega(x_0)$$

$$\alpha\text{-limit set } \mathcal{L}_\alpha := \bigcup_{x_0 \in \mathbb{R}^n} \alpha(x_0)$$

Theorem 3

$\bar{y} \in \mathbb{R}^n$ is an ω -limit point or α -limit point of $\dot{x} = -\nabla V(x)$, then $\bar{y}$ is an EP of V .

Proof Let $\bar{y} \in \mathcal{L}_\omega$. Suppose $\bar{y}$ not an EP. Then $\nabla V(\bar{y}) \neq 0$, and $\lim_{n \rightarrow \infty} \phi(t_n, x_0) = \bar{y}$ for some x_0 and $t_n \xrightarrow{n \uparrow \infty} \infty$.

$$\text{As } \frac{d}{dt} V(\phi(t, x_0)) = -|\nabla V(\phi(t, x_0))|^2 \leq 0,$$

$$MCT \Rightarrow V(\phi(t, x_0)) \downarrow V(\bar{y}) \text{ as } t \rightarrow \infty \quad (5).$$

As $\omega(\Sigma_0)$ is positively invariant,
 $\phi(t, \Sigma) \in \omega(\Sigma_0) \quad \forall t \geq 0$, and
 $\frac{d}{dt} V(\phi(t, \Sigma))|_{t=0} = -|\nabla V(\Sigma)|^2 < 0$.

$$\Rightarrow V(\phi(t, \Sigma)) < V(\Sigma) \quad \forall t > 0.$$

Recalling $\phi(t, \Sigma) \stackrel{(4)}{=} \lim_{n \rightarrow \infty} \phi(t + t_n, \Sigma_0)$, this
 leads to contradiction

$$\lim_{t \rightarrow \infty} V(\phi(t, \Sigma)) = \lim_{t \rightarrow \infty} \left(\lim_{n \rightarrow \infty} V(\phi(t + t_n, \Sigma_0)) \right) \\ \stackrel{(5)}{=} V(\Sigma).$$

If instead $\Sigma \in \mathcal{L}_\alpha$, then $\Sigma \in \mathcal{L}_\omega$ for

$$\tilde{\Sigma}' = \nabla V(\tilde{\Sigma}) \quad (\tilde{\Sigma}(t) := \Sigma(-t))$$

and same argument applies.

