

Lecture 24

Consider

$$\mathbf{X}' = F(\mathbf{X}, t)$$

$$\mathbf{X}(0) = \bar{\mathbf{X}}_0 \in \mathbb{R}^d$$

When $F \in C^1(\mathbb{R}^d \times [0, T], \mathbb{R}^d)$ is globally Lipschitz in $\mathbf{X}$ & t on $\mathbb{R}^d \times [0, T]$, and a method $\bar{\mathbf{X}}_{n+1} = \bar{\mathbf{X}}_n + \Phi(\bar{\mathbf{X}}_n, \bar{\mathbf{X}}_{n+1}, t_n, h)h$ is stable: $\max_{0 \leq n \leq N} |\bar{\mathbf{X}}_n| \leq C_2 |\bar{\mathbf{X}}_0| + C_1$

with $t_n = n \cdot h$, $h = \frac{T}{N}$, then

$$|\mathbf{X}(T) - \bar{\mathbf{X}}_N| \leq C_\Phi \sum_{n=0}^{N-1} |\tau(\bar{\mathbf{X}}_n, t_n)|$$

where

$$\tau(\bar{\mathbf{X}}_n, t_n) = \mathbf{X}(t_{n+1}; \mathbf{X}(t_n) = \bar{\mathbf{X}}_n) - \bar{\mathbf{X}}_{n+1}$$

$$= \bar{\mathbf{X}}_n + \int_{t_n}^{t_{n+1}} F(\mathbf{X}(s; \mathbf{X}(t_n) = \bar{\mathbf{X}}_n), s) ds$$

$$- \bar{\mathbf{X}}_n - \Phi(\bar{\mathbf{X}}_n, \bar{\mathbf{X}}_{n+1}, t_n, h)h$$

$$= \int_{t_n}^{t_{n+1}} F(\mathbf{X}(s; \bar{\mathbf{X}}_n), s) - \Phi(\bar{\mathbf{X}}_n, \bar{\mathbf{X}}_{n+1}, t_n, h) ds$$

If $|\tau(\bar{x}_n)| \leq Ch^{1+p}$ for $p > 0$, then

$|\underline{x}(T) - \bar{x}_N| \leq Ch^p$ and p is the order of the method.

Consistency $p > 0$ holds only if

$$\lim_{h \rightarrow 0} \Phi(\bar{x}_n, \bar{x}_{n+1}, t_n, h) = F(\bar{x}_n, t_n).$$

1-st order methods: Forward Euler

$$\bar{x}_{n+1} = \bar{x}_n + \underbrace{F(\bar{x}_n, t_n)}_{\Phi(\bar{x}_n, t_n)} h$$

$$\Rightarrow \tau(\bar{x}_n, t_n) = \int_{t_n}^{t_{n+1}} F(\underline{x}(s; \bar{x}_n), s) - F(\bar{x}_n, t_n) ds$$

$$= \int_{t_n}^{t_{n+1}} \int_{t_n}^s \underbrace{\frac{d}{dt} F(\underline{x}(t; \bar{x}_n), t)}_{\leq C \text{ in norm}} dt ds$$

$$\Rightarrow |\tau(\bar{x}_n, t_n)| \leq C \frac{h^2}{2}$$

Implicit Euler:

$$\bar{X}_{n+1} = \bar{X}_n + \underbrace{F(\bar{X}_{n+1}, t_n)}_{\Phi(t_n, \bar{X}_n)} h$$

also has $p=1$.

2nd order methods : Idea

$$X(t_{n+1}) - X(t_n) = \int_{t_n}^{t_{n+1}} F(X(s), s) ds$$

$$\approx \Phi(X(t_n), X(t_{n+1}), t_n, h) h$$

2nd order quadrature:

$$\int_{t_n}^{t_{n+1}} F(X(s), s) ds = \frac{h}{2} (F(X(t_n), t_n) + F(X(t_{n+1}), t_{n+1})) + \mathcal{O}(h^3)$$

CN-method (Implicit method):

2-stages $k_1 = F(\bar{X}_n, t_n)$, $k_2 = F(\bar{X}_{n+1}, t_{n+1})$

$$\bar{X}_{n+1} = \underbrace{\frac{1}{2}(k_1 + k_2)}_{\Phi(\bar{X}_n, \bar{X}_{n+1}, t_n, h)} h$$

Heun's method (Explicit method):

2-stages

$$k_1 = F(\bar{x}_n, t_n)$$

$$k_2 = F(\bar{x}_n + h k_1, t_n + h)$$

$$\bar{x}_{n+1} = \bar{x}_n + \underbrace{\frac{k_1 + k_2}{2}}_{\Phi(\bar{x}_n, t_n, h)} h$$

(Same order as CN follows from

$$k_2 = F(\bar{x}_{n+1}, t_{n+1}) + O(h^2))$$

Modified Euler

Based on midpoint quadrature

$$\int_{t_n}^{t_{n+1}} F(x(s), s) ds = F\left(x\left(t_n + \frac{h}{2}\right), t_n + \frac{h}{2}\right) \frac{h}{2} + O(h^3)$$

and 2-stages $k_1 = F(\bar{x}_n, t_n)$

$$k_2 = F\left(\bar{x}_n + \frac{h}{2} k_1, t_n + \frac{h}{2}\right)$$

$$\bar{X}_{n+1} = \bar{X}_n + K_2 \cdot h$$

Runge kutta methods

Any s-stage one-step method where Φ can be represented by

$$\begin{array}{c|ccc} c_1 & a_{11} & \dots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \dots & a_{ss} \\ \hline & b_1 & \dots & b_s \end{array}$$

$$\left(\text{compactly } \frac{C}{b^T} \right)$$

$$\text{where } K_i = F\left(\bar{X}_n + h \sum_{j=1}^s a_{ij} K_j, t_n + c_i h\right)$$

$$\text{and } \bar{X}_{n+1} = \bar{X}_n + h \sum_{i=1}^s b_i K_i$$

Forward Euler: $s=1$

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

$$K_1 = F(\bar{X}_n, t_n)$$

$$\bar{X}_{n+1} = \bar{X}_n + h \cdot K_1$$

Implicit Euler

1		1

$$K_1 = F(\bar{X}_n + h a_{11} k_1, t_n + c_1 h)$$

$$= F(\bar{X}_n + h k_1, t_{n+1}) = F(\bar{X}_{n+1}, t_{n+1})$$

$$\begin{aligned}\bar{X}_{n+1} &= \bar{X}_n + h b_1 k_1 = \bar{X}_n + h k_1 \\ &= \bar{X}_n + h F(\bar{X}_{n+1}, t_{n+1})\end{aligned}$$

CN method

2-stage method

0	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$
	$\frac{1}{2}$	$\frac{1}{2}$

$$K_1 = F(\bar{X}_n, t_n)$$

$$K_2 = F(\bar{X}_n + \frac{h}{2}(K_1 + K_2), t_n + h)$$

$$\bar{X}_{n+1} = \bar{X}_n + h\left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)$$

$$\Rightarrow k_2 = F(\bar{X}_{n+1}, t_{n+1}) \text{ and thus}$$

$$\bar{X}_{n+1} = \underbrace{\bar{X}_n + \frac{h}{2}(F(\bar{X}_n, t_n) + F(\bar{X}_{n+1}, t_{n+1}))}_{\text{The definition of the CN method}}$$

The definition of the CN method

can show similarly that

$$\text{Heun's method} = \begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

$$\text{Modified Euler} = \begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array}$$

RK4 Is an explicit $s=4$ stage method based on Simpson's rule approx of

$$\int_{t_n}^{t_{n+1}} F(\bar{X}(s), s) ds.$$

$$K_1 = F(\bar{X}_n, t_n)$$

$$K_2 = F(\bar{X}_n + \frac{h}{2} K_1, t_n + \frac{h}{2})$$

$$K_3 = F(\bar{X}_n + \frac{h}{2} K_2, t_n + \frac{h}{2})$$

$$K_4 = F(\bar{X}_n + h K_3, t_n + h)$$

0	0	0	0	0
1/2	1/2	0	0	0
1/2	0	1/2	0	0
1	0	0	1	0
	1/6	2/6	2/6	1/6

$$\bar{X}_{n+1} = \bar{X}_n + \underbrace{\frac{K_1 + 2K_2 + 2K_3 + K_4}{6}}_{\Phi(\bar{X}_n, t_n, h)} h$$

Properties

- Consistency $\left(\lim_{h \rightarrow 0} \Phi(\bar{x}_n, \dots, h) = F(\bar{x}_n, t) \right)$ requires that $\sum_{i=1}^s b_i = 1$
- $A \in \mathbb{R}^{s \times s}$ strictly lower triangular $\Rightarrow$ RK method is explicit. Otherwise, it is implicit.
- For $s \geq 2$, one can show that a consistent method has order $p \in [2, 2s]$ if also

$$C_i = \sum_{j=1}^s a_{ij} \quad i=1, \dots, s \quad \text{and}$$

$$\sum_{i=1}^s b_i C_i = \frac{1}{2}$$

and $p \leq 2$ if the method is explicit.

Motivation, take $d=1$:

$$K_i = f(\bar{x}_n + h \sum_j a_{ij} K_j, t_n + C_i h)$$

$$= f(\bar{x}_n, t_n) + h \left(a_i f_t(\bar{x}_n, t_n) + \underbrace{\sum_j a_{ij} K_j f_x(\bar{x}_n, t_n)}_{= C_i} \right) + O(h^2)$$

$$\text{Then } \Phi(\bar{x}_n, \bar{x}_{n+1}, t_n, h) = \sum_{i=1}^s b_i K_i =$$

$$f(\bar{x}_n, t_n) + h^2 \sum_{i=1}^s b_i c_i (f_t + f_x f)(\bar{x}_n, t_n) + \mathcal{O}(h^3)$$

$$\Rightarrow \tau(\bar{x}_n, t_n) = \mathcal{O}(h^2).$$