

Week 16 Exercise 1

a) Show that if $\bar{x}^*$ is A-stable for $\bar{x}' = F(\bar{x})$, then $\bar{x}^*$ is unstable for $\bar{y}' = -F(\bar{y})$.

b) Show that if there for $\bar{x}' = F(\bar{x})$ with EP $\bar{x}^*$ exists an L s.t.

$$L(\bar{x}^*) = 0, \quad L(\bar{x}) > 0 \text{ and } \dot{L}(\bar{x}) > 0$$

for all $\bar{x}$ near $\bar{x}^*$, then $\bar{x}^*$ is unstable.

Sol a) For sufficiently small $\delta > 0$, take an $\bar{x}_0 \in \partial B_\delta(\bar{x}^*)$ and sequence $t_n \rightarrow \infty$ s.t. $|\bar{x}(t_n) - \bar{x}^*| = 2^{-n} \delta$.

Note: If $\bar{x}$ solves $\bar{x}' = F(\bar{x})$, then

$\bar{y}(t) = \bar{x}(-t)$ solves $\bar{y}' = -F(\bar{y})$.

(as $\bar{y}'(t) = -\bar{x}'(-t) = -F(\bar{x}(-t)) = -F(\bar{y})$)

So $\bar{y}(t; \bar{x}_0) = \bar{x}(-t; \bar{x}_0) = \phi(-t, \bar{x}_0)$.

Considering the sequence $\underline{X}(t_n)$, note,
 $\underline{Y}(t_n; \underline{X}(t_n)) = \phi(-t_n, \underline{X}(t_n)) = \underline{X}_0 \quad \forall n$.

Hence, we can find a point $\underline{X}(t_n)$
arbitrarily close to $\underline{X}^*$ and a $t = t_n$
s.t. $|\underline{Y}(t_n; \underline{X}(t_n)) - \underline{X}^*| \geq d$.

b) Consider $\underline{Y}' = -\underline{F}(\underline{Y})$. For this
dynamics, L is a strict Lyapunov
fcn for $\underline{X}^*$ as
$$\dot{L}(\underline{X}) = \nabla L \cdot (-\underline{F}) = -\underbrace{(\nabla L \cdot \underline{F})}_{>0} < 0.$$

for any $\underline{X}$ near $\underline{X}^*$.

Hence $\underline{X}^*$ is A-stable for $\underline{Y}$.

Now $\underline{X}(t) := \underline{Y}(-t)$ solves
 $\underline{X}'(t) = \underline{F}(\underline{X})$, so by a) $\underline{X}^*$
is unstable for $\underline{X}$.

Exercise 2 Week 16

Show that for

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1^3 - x_2$$

the EP $\bar{x}^* = 0$ is stable using

$$L(\bar{x}) = \frac{x_1^4}{4} + \frac{x_2^2}{2} + \epsilon x_1^3 x_2$$

for a small $\epsilon > 0$.

Sol Let $\epsilon = \frac{1}{10}$ and $D = B_1(0)$.

$L(0) = 0$, and

$$\epsilon |x_1^3 x_2| \leq \epsilon \left(\frac{x_1^4}{2} + \frac{x_1^2 x_2^2}{2} \right)$$

$$\Rightarrow \epsilon x_1^3 x_2 \geq -\epsilon \left(\frac{x_1^4}{2} + \frac{x_1^2 x_2^2}{2} \right), \text{ so}$$

$$L(\bar{x}) \geq \frac{x_1^4}{4} (1 - 2\epsilon) + \frac{x_2^2}{2} (1 - \epsilon x_1^2)$$

$$\geq \frac{x_1^2 + x_2^2}{5}$$

$$\forall \bar{x} \in B_1(0)$$

Which means $L(\underline{x}) > 0 \quad \forall \underline{x} \in B_1(0) \setminus \{0\}$.

$$\text{Next, } \dot{L}(\underline{x}) = \begin{bmatrix} x_1^3 + 3\epsilon x_1^2 x_2 \\ x_2 + \epsilon x_1^3 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ -x_1^3 - x_2 \end{bmatrix}$$

$$= \cancel{x_1^3 x_2} + 3\epsilon x_1^2 x_2^2 - \cancel{x_2 x_1^3} - x_2^2 \\ - \epsilon x_1^6 - \epsilon x_1^3 x_2$$

$$\leq 3\epsilon x_1^2 x_2^2 - x_2^2 - \epsilon x_1^6 + \frac{\epsilon}{2} x_1^6 + \frac{\epsilon}{2} x_2^2$$

$$= -x_2^2 \underbrace{\left(1 - \frac{\epsilon}{2} - 3\epsilon x_1^2\right)}_{> \frac{4}{5}} - \frac{\epsilon x_1^6}{2}$$

$$\Rightarrow \dot{L}(\underline{x}) \leq \frac{-x_2^2 - x_1^6}{20} \quad \underline{x} \in B_1(0)$$

$$\Rightarrow \dot{L}(\underline{x}) < 0 \quad \forall \underline{x} \in B_1(0) \setminus \{0\}.$$

Have shown that L is strict Lyapunov for 0, so 0 is A-stable.

9.4 Consider

$$x' = (\epsilon x + 2y)(z+1)$$

$$y' = (-x + \epsilon y)(z+1)$$

$$z' = -z^3$$

a) show when $\epsilon = 0$ then 0 is not A-stable

b) show when $\epsilon < 0$, the basin of attraction contains $\{(x, y, z) \mid z > -1\}$.

sol a) $x' = 2yz + 2y$

$$y' = -xz - x$$

$$z' = -z^3$$

$$\text{Let } V(\mathbb{X}) = \frac{x^2}{2} + y^2$$

$$\frac{d}{dt} V(\mathbb{X}(t)) = xx' + 2yy' = 0$$

$\Rightarrow$ any solution with $V(\mathbb{X}(0)) = \alpha > 0$,
will satisfy $V(\mathbb{X}(t)) = \alpha \quad \forall t \geq 0$

$\Rightarrow \mathbb{X}(t; \mathbb{X}_0) \not\rightarrow 0$ for any

$\mathbb{X}_0 = (x_0, y_0, z_0)$ with x_0 or $y_0 \neq 0$.

b) Find strict Lyapunov fun L on

$$D \supset \{(x, y, z) \mid z > -1\}. \text{ Let}$$

$$L(\underline{x}) = \frac{x^2}{2} + y^2 + z^2$$

$$\Rightarrow \dot{L}(\underline{x}) = \nabla L \cdot F = \begin{bmatrix} x \\ 2y \\ 2z \end{bmatrix} \cdot \begin{bmatrix} (\epsilon x + 2y)(z+1) \\ (-x + \epsilon y)(z+1) \\ -z^3 \end{bmatrix}$$

$$= \epsilon(x^2 + 2y^2)(z+1) - 2z^4$$

$$< 0 \quad \forall \underline{x} \in \{(x, y, z) \mid z > -1\}$$

when $\epsilon < 0$.

9.6 Find a strict Lyapunov fcn for the EPO of

$$x' = -2x - y^2$$

$$y' = -y - x^2$$

Find $\delta > 0$ as large as possible s.t.

$$x_0 \in B_\delta(0) \Rightarrow x(t; x_0) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Take

Sol $L(x) = \alpha \frac{x^2}{2} + \beta \frac{y^2}{2}$ for $\alpha, \beta > 0$.

$$L(0) = 0, L(x) > 0 \text{ for } x \neq 0 \text{ ok.}$$

$$\dot{L}(x) = \nabla L \cdot F = \begin{bmatrix} \alpha x \\ \beta y \end{bmatrix} \cdot \begin{bmatrix} -2x - y^2 \\ -y - x^2 \end{bmatrix}$$

$$= -2\alpha x^2 - \alpha x y^2 - \beta y^2 - \beta y x^2$$

$$= -x^2(2\alpha - \beta y) - y^2(\beta - \alpha x) < 0$$

$$\text{if } y < \frac{2\alpha}{\beta} \text{ and } x < \frac{\beta}{\alpha}$$

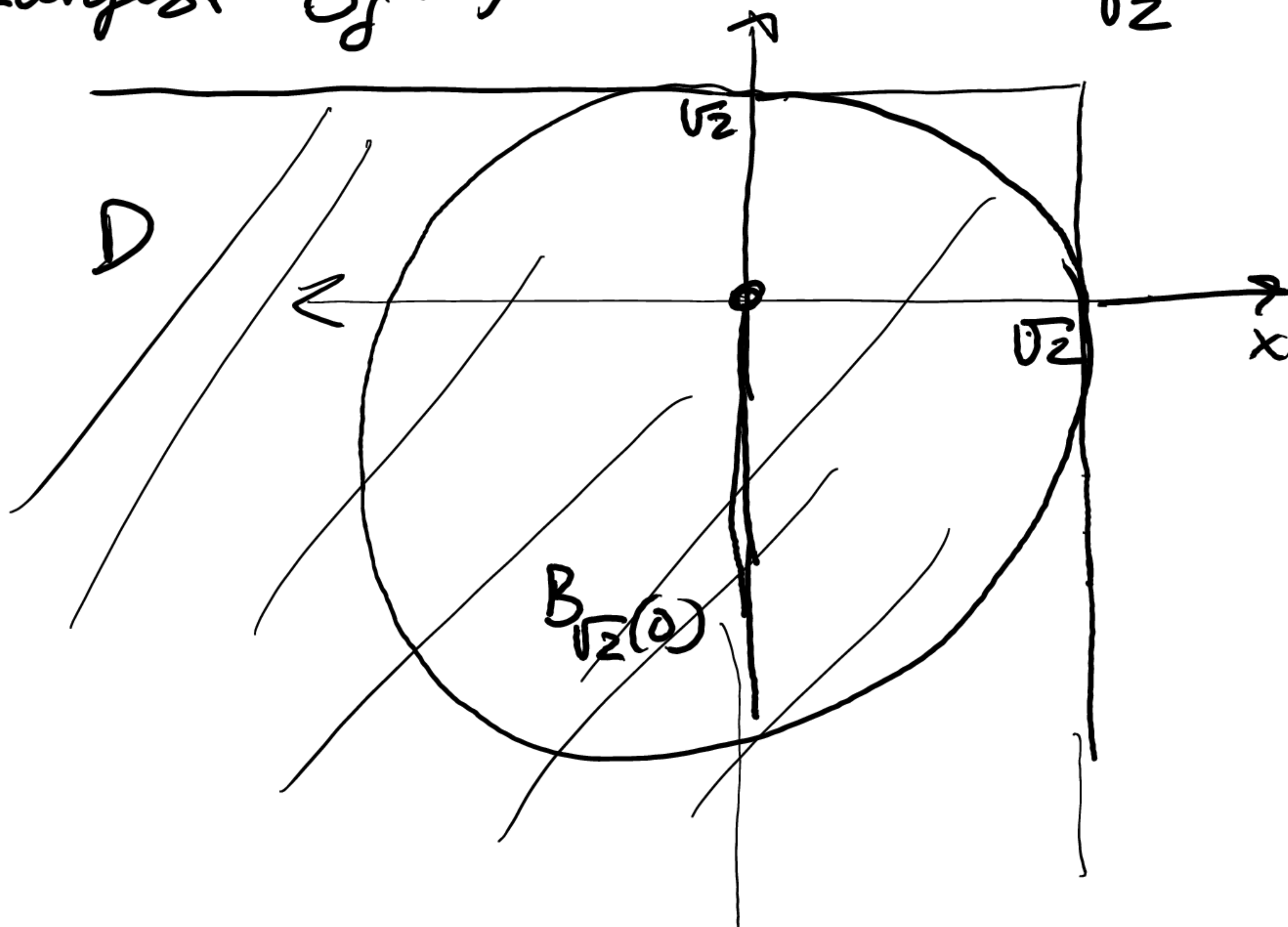
equilibrate constraints to maximize

$$\delta: \frac{2\alpha}{\beta} = \frac{\beta}{\alpha} \Rightarrow \alpha = \frac{\beta}{\sqrt{2}}$$

Leads to

$$L(\mathbb{R}) < \infty \quad \text{for } \underbrace{\{(x, y) \mid \max(x, y) < \sqrt{2}\}}_D$$

Largest $B_\delta(0)$ in D is $B_{\sqrt{2}}(0)$.



Week 17 Exercise 1

a) Show that if a continuous $i: [0, \infty) \rightarrow [0, \infty)$ satisfies

$$i(t) \leq C + Mt + L \int_0^t i(s) ds \quad t \geq 0$$

for some $C, M, L > 0$, then

$$i(t) \leq e^{Lt} C + \frac{M}{L} (e^{Lt} - 1).$$

(Hint: Extend proof of Grönwall's inequality in Lecture 17).

b) Let $\mathbf{x}' = F(\mathbf{x})$ where

$F \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ with global Lipschitz constant $L > 0$. Show that

$$|\phi(t, \mathbf{x}_0)| \leq e^{Lt} \left(|\mathbf{x}_0| + \frac{\|F(0)\|}{L} \right)$$

for any $t > 0$ and $\mathbf{x}_0 \in \mathbb{R}^d$

Hint: $|F(\mathbf{x}) - F(0)| \leq L|\mathbf{x}|$

Sol

a) Let $I(t) = \int_0^t i(s) ds$. Then

$$I'(t) \leq C + Mt + LI$$

$$\frac{d}{dt}(e^{Lt} I(t)) \leq (C + Mt)e^{-Lt} / \int_0^s$$

$$\Rightarrow I(s)e^{-Ls} \leq I(0) + \int_0^s (C + Mt)e^{-Lt} dt$$

$$= -\frac{1}{L} \left[(C + Mt)e^{-Lt} \right]_0^s - \frac{M}{L^2} \left[e^{-Lt} \right]_0^s$$

$$\leq \left(\frac{C}{L} + \frac{M}{L^2} \right) (1 - e^{-Ls}) - \frac{Ms}{L} e^{-Ls}$$

$$\Rightarrow I(s) \leq \left(\frac{C}{L} + \frac{M}{L^2} \right) (e^{Ls} - 1) - \frac{Mt}{L}$$

$$\Rightarrow i(t) \leq C + Mt + LI(t)$$

$$\leq Ce^{Lt} + \frac{M}{L}(e^{Lt} - 1).$$

$$b) \quad X(t) = X_0 + \int_0^t F(X(s)) ds$$

$$\Rightarrow |X(t)| \leq |X_0| + \int_0^t |F(X(s))| ds$$

$$\leq |x_0| + \int_0^t |F(x(s)) - F(0)| + |F(0)| ds$$

$$\leq |x_0| + |F(0)|t + L \int_0^t |x(s)| ds$$

Let $c = |x_0|$, $M = |F(0)|$ and

$i(t) = |x(t)|$, then

$$|x(t)| \leq c e^{Lt} + \frac{|F(0)|}{L} (e^{Lt} - 1)$$

by a).

Ex 9.7 Find EP for $x' = -\nabla V(x)$ and determine type:

b) $V(x) = x^2 - y^2 - 2x + 4y + 5$

c) $V(x) = y \sin x$

f) $V(x) = x^2(x-1) + y^2(y-2) + z^2$

Sol b) $V(x) = (x-1)^2 - (y-2)^2$

$$\nabla V(x) = 2 \begin{bmatrix} x-1 \\ 2-y \end{bmatrix}$$

$$\text{EP: } x^* = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$F(\underline{x}) = -\nabla V(\underline{x}) \quad \&$$

$$DF(\underline{x}^*) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \underline{x}^* \text{ is saddle.}$$

$$c) -\nabla V(\underline{x}) = \begin{bmatrix} -y \cos x \\ -\sin x \end{bmatrix} =: F(\underline{x})$$

$$EP: (k\pi, 0) \quad k \in \mathbb{Z}$$

$$DF(\underline{x}) = \begin{bmatrix} y \sin x & -\cos x \\ -\cos x & 0 \end{bmatrix}$$

$$DF\left(\begin{bmatrix} k\pi \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & (-1)^{k+1} \\ (-1)^{k+1} & 0 \end{bmatrix}$$

$$\text{eigenvalues } \lambda_{\pm} = \pm 1 \Rightarrow \begin{bmatrix} k\pi \\ 0 \end{bmatrix} \text{ is saddle.}$$

$$f) V(\underline{x}) = x^2(x-1) + y^2(y-2) + z^2$$

$$\Rightarrow -\nabla V(\underline{x}) = -\begin{bmatrix} x(3x-2) \\ y(3y-4) \\ 2z \end{bmatrix} =: F(\underline{x})$$

$$EP: 0, \begin{bmatrix} 2/3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4/3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 4/3 \\ 0 \end{bmatrix}$$

$\bar{x}_1^* \quad \bar{x}_2^* \quad \bar{x}_3^* \quad \bar{x}_4^*$

Linearization:

$$DF(\bar{x}) = \begin{bmatrix} 2-6x & 0 & 0 \\ 0 & 4-6y & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$\bar{x}_1^*$ - saddle ($\lambda_1, \lambda_2 > 0, \lambda_3 < 0$)

$\bar{x}_2^*$ - saddle ($\lambda_2 > 0, \lambda_1, \lambda_3 < 0$)

$\bar{x}_3^*$ - saddle ($\lambda_1 > 0, \lambda_2, \lambda_3 < 0$)

$\bar{x}_4^*$ - sink ($\lambda_1, \lambda_2, \lambda_3 < 0$)

($\bar{x}_4^*$ is the global minimum).

9.9. a) Let $\bar{x}' = A\bar{x}$ be a linear system where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Determine conditions on a, b, c, d that guarantee that it is a gradient system.

$$a) \dot{X} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

as system is linear (degree 1 polynomial),
can only have

$$V(X) = c_1 x^2 + c_2 xy + c_3 y^2 + c_4$$

yields equations

$$-\partial_x V = ax + by \Rightarrow -2c_1 = a \text{ \& } -c_2 = b$$

$$-\partial_y V = cx + dy \Rightarrow -c_2 = c \text{ \& } -2c_3 = d$$

that have a solution iff $c = b$

Hence, any symmetric $A \in \mathbb{R}^{2 \times 2}$
associated to / is a gradient system

$$\dot{X} = AX = -\nabla V(X) \text{ where}$$

$$V(X) = -\frac{a}{2}x^2 - bxy - \frac{d}{2}y^2$$

q. 10 consider

$$x' = f(x, y)$$

$$y' = g(x, y)$$

Determine conditions on f and g that ensure that it is a gradient system

sol

$$V_x(x, y) = -f(x, y)$$
$$V_y(x, y) = -g(x, y)$$

suppose $V \in C^2$. Then $V_{xy} = V_{yx}$.

Hence, must have $f_y = g_x$.

Consider next

$$V(x, y) = \int_0^y \int_0^x -g_x(r, s) dr ds - \int_0^y g(0, s) ds \\ - \int_0^x f(r, 0) dr$$

Suppose $f_y = g_x$. Observe that then

$$\text{Then } V_y(x, y) = \int_0^x -g_x(r, y) dr - g(0, y) \\ = -g(x, y)$$

$$\text{and, as } \int_0^y \int_0^x -g_x(r, s) dr ds = \int_0^x \int_0^y -f_y(r, s) ds dr,$$

$$V_x(x, y) = \int_0^y -f_y(x, s) ds - f(x, 0) \\ = -f(x, y).$$

Conclusion if $f_y = g_x$, then

$\begin{matrix} x' = f \\ y' = g \end{matrix}$ is a gradient system.

This corresponds to $DF = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$

being symmetric for every (x, y)

(as extension of result in 9.9).