

Lecture 26

Stiff problems

Is a property of a given IVP and numerical method.

$$\text{If } \begin{aligned} \underline{X}' &= F(\underline{X}, t) & t \in [0, T] \\ \underline{X}(0) &= \underline{X}_0 \in \mathbb{R}^d \end{aligned}$$

satisfies $|\underline{X}(t)| \leq C_1 |\underline{X}_0| + C_2$

for small $C_1, C_2 \geq 0$, but (num solution)

$|\bar{X}_n|$ grows fast in n for

large values of $h > 0$, then the

IVP + num method is called

stiff.

Ex 1

$$x' = -10^6 x \quad t \in [0, 10]$$

$$x(0) = 1$$

$$x(t) = e^{-10^6 t}$$

Forward Euler $h = \frac{1}{10}$.

$$x_{n+1} = (1 - 10^6 \cdot h) x_n = (1 - 10^5)^{n+1} x_0$$

$$\Rightarrow |x_n| \approx 10^{5n}$$

IVP + Forward Euler: a stiff problem.

Compare to implicit Euler

$$x_n = \frac{1}{(1 + 10^6 h)^n} x_0 \quad \Rightarrow |x_n| \leq 1 \quad \forall \begin{pmatrix} n \\ h > 0 \end{pmatrix}$$

IVP + implicit Euler: a non-stiff problem.

Ex 2 (HSD 7.6)

$$x' = e^t \sin x \quad t \in [0, 10]$$

$$x(0) = 0.3$$

$$x(t) = 2 \tan^{-1} \left(\tan\left(\frac{x_0}{2}\right) e^{e^t - 1} \right)$$

$$\text{with } \sup_{t \geq 0} |x(t)| = \pi$$

Will see numerically that FE

$$\bar{x}_{n+1} = \bar{x}_n + (e^{t_n} \sin \bar{x}_n) h \quad \text{for } h > 10^{-4}$$

leads to $|\bar{x}_n|$ growing quickly

(stiff problem).

(IE and CN have bounded solutions even for "large" $h > 0$.)

Linear stability

We consider $\underline{X}' = A\underline{X}$, $A \in \mathbb{R}^{n \times n}$
and say that the ODE is linearly
stable if for any $\underline{X}_0 \in \mathbb{R}^n$, $\exists C > 0$
s.t. $|\underline{X}(t)| \leq C |\underline{X}_0| \quad \forall t \geq 0$.

As $\underline{X}(t) = e^{At} \underline{X}_0$, this holds
iff $\operatorname{re}(\lambda) \leq 0$ for all EV λ of A .

Q: Which Runge-Kutta methods are
stable: $|\bar{\underline{X}}_n| \leq C_1 |\underline{X}_0| + C_2 \quad \forall n$
when applied to linearly stable ODE?

1D setting

Consider $x' = \lambda x$, for $\lambda = a + ib \in \mathbb{C}$

$$x(t) = e^{\lambda t} x_0 = e^{at} e^{ibt} x_0 \Rightarrow$$

ODE is linearly stable iff $a \leq 0$.

We write

$$\lambda \in \mathbb{C}_- = \{z \in \mathbb{C} \mid \operatorname{Re}(z) \leq 0\} \Leftrightarrow \text{linearly stable!}$$

Result: For any RK method applied to $x' = \lambda x$, we can find a stability function s.t.

$$\bar{x}_{n+1} = R(\lambda h) \bar{x}_n \quad \text{where}$$

$$R(z) = \frac{p(z)}{q(z)} \quad \text{is a rational function.}$$

And $q(z) = 1$ when method is explicit.

Examples

$$\text{FE: } \bar{x}_{n+1} = \bar{x}_n + \lambda \bar{x}_n h = (1 + \lambda h) \bar{x}_n$$

$$R(\lambda h) = 1 + \lambda h$$

$$\text{IE: } \bar{x}_{n+1} = \frac{1}{1 - \lambda h} \bar{x}_n, \quad R(\lambda h) = (1 - \lambda h)^{-1}$$

(well defined when $\lambda < 0$)

$$\text{CN: } \bar{x}_{n+1} = \underbrace{\frac{(1+2h/2)}{(1-2h/2)}}_{R(2h)} \bar{x}_n$$

Given $x' = \lambda x$, $\lambda \in \mathbb{C}$ the solution of an RK method is

$$\bar{x}_n = R(2h) \bar{x}_{n-1} \Rightarrow |\bar{x}_n| = |R(2h)|^n |\bar{x}_0|.$$

Writing $z = 2h \in \mathbb{C}$, we conclude that

$$|\bar{x}_n| \text{ is } \underbrace{\text{bounded } \forall (n, x_0)}_{\text{stable}} \Leftrightarrow |R(z)| \leq 1.$$

Def An RK method with stability function $R: \mathbb{C} \rightarrow \mathbb{C}$ has the stability region

$$S = \{z \in \mathbb{C} \mid |R(z)| \leq 1\}.$$

A method is called absolutely stable if $\mathbb{C}_- := \{z \mid \operatorname{Re}(z) \leq 0\} \subset S$.

Note $z = \lambda h \in \mathbb{C}_- \Rightarrow \operatorname{Re}(\lambda) \leq 0$,
so absolutely stable means that
 $|\bar{X}_n|$ is bounded for any $h > 0$ when $\operatorname{Re}(\lambda) \leq 0$.

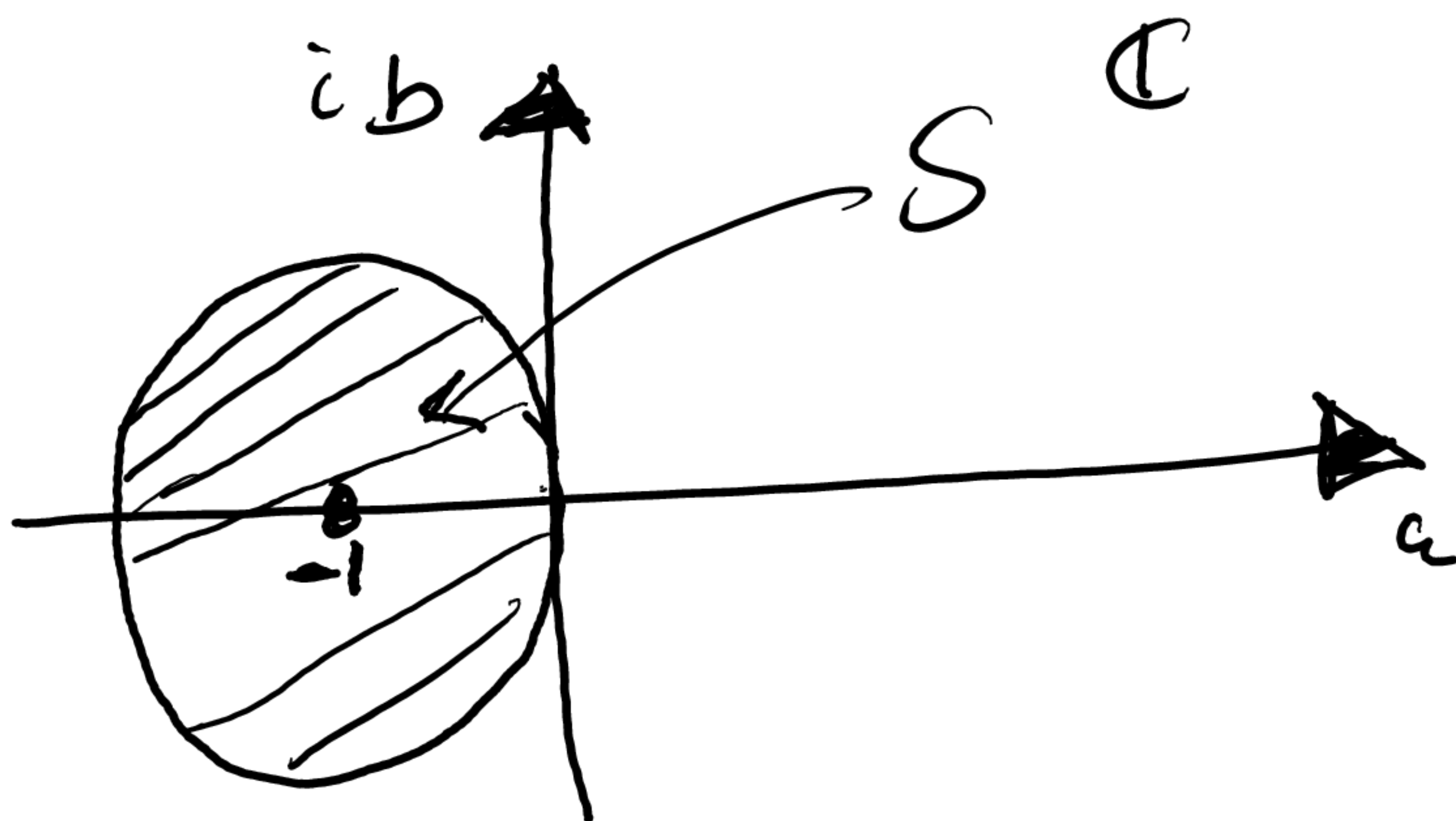
Stability regions

$$FE: R(\lambda h) = 1 + \lambda h$$

$$|R(z)| \leq 1 \Leftrightarrow |1+z| \leq 1$$

$$(z = a + ib)$$

$$\Leftrightarrow (1+a)^2 + b^2 \leq 1$$



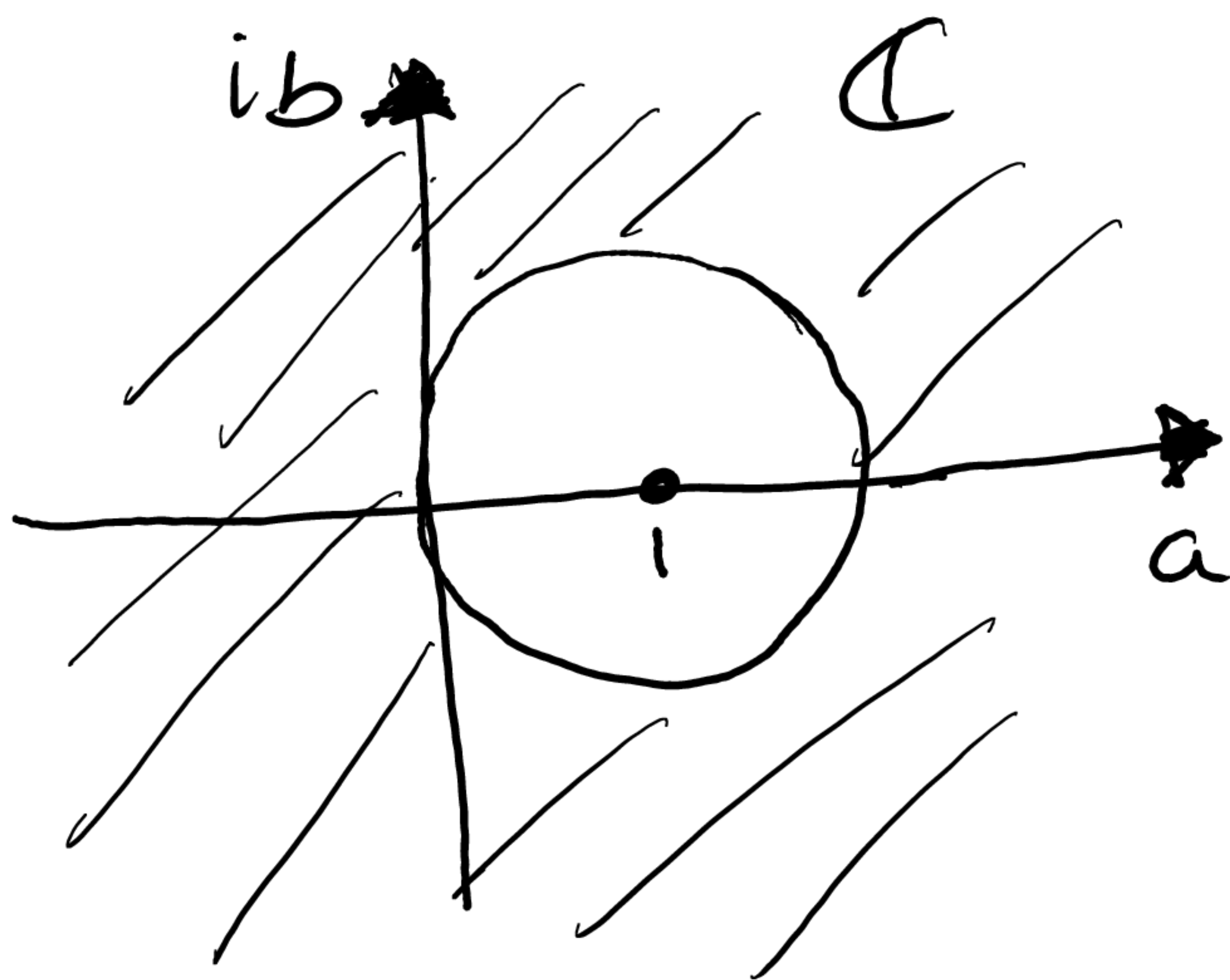
As $S \subsetneq \mathbb{C}$, FE is not absolutely stable.

(Need $h\lambda \in S$ to ensure stability.)

$$IE: R(z) = (1-z)^{-1}$$

$$|R(z)| \leq 1 \Leftrightarrow 1 \leq |1-z|$$

$$\Leftrightarrow (1-a)^2 + b^2 \geq 1$$



$$\boxed{//} = S$$

here $S \supset \mathbb{C}_-$ so IE is absolutely stable.

Whenever $\operatorname{Re}(\lambda) \leq 0$, then
IE is stable for any $h > 0$, as
 $\lambda h \in S$

$$\text{CN: } R(z) = \frac{1 + z/2}{1 - z/2}$$

$$|R(z)| \leq 1 \Leftrightarrow |1 + z/2|^2 \leq |1 - z/2|^2$$

$$\Leftrightarrow (1 + a/2)^2 + (b/2)^2 \leq (1 - a/2)^2 + (b/2)^2$$

$$\Leftrightarrow a \leq 0. \quad \text{so } S = \mathbb{C}_-$$

