

lecture 27

Classical mechanics

Let $\underline{x}(t) \in \mathbb{R}^d$ be position of a particle
with mass m ($d \in \{1, 2, 3\}$)

Newton's 2nd law:

$$\underbrace{F}_{\text{Force}} = m \underline{x}''(t)$$

mass $\times$ acceleration

We will mainly consider force fields

$$\underbrace{F}_m = -\nabla U(\underline{x}) \text{ for a } \underline{\text{potential}} \quad U: \mathbb{R}^d \rightarrow \mathbb{R}.$$

Ex 1 $F = g = -9.81 \text{ m/s}^2 \quad (m=1)$

$$\Rightarrow \underline{z}'' = -g = -u'(z) \quad \text{with } u(z) = gz.$$

Problem

$$\underline{x}'' = -\nabla U(\underline{x})$$

writing $V := \underline{x}'$ yields system

$$\left. \begin{aligned} \underline{x}' &= V \\ V' &= -\nabla U(\underline{x}). \end{aligned} \right\} (1)$$

Conservation of Hamiltonian

Second order eq. yields

$$\underline{x}' \cdot (\underline{x}'' + \nabla U(\underline{x})) = 0 \quad | \underline{x}' \cdot$$

can rewrite as

$$\frac{d}{dt} \left(\underbrace{\frac{|\underline{x}'|^2}{2} + U(\underline{x})}_{\frac{|V|^2}{2} + U(\underline{x})} \right) = 0$$

Def $H(\underline{x}, v) := \underbrace{\frac{|v|^2}{2}}_{\text{kinetic energy}} + \underbrace{U(\underline{x})}_{\text{potential energy}}$

is the Hamiltonian of ODE (1),

and it is a conserved quantity along solution paths, meaning

$$\frac{d}{dt} H(\underline{x}(t), v(t)) = 0.$$

This means that (1) can be written as a Hamiltonian system

$$\underline{x}' = \nabla_v H(\underline{x}, v)$$

$$v' = -\nabla_{\underline{x}} H(\underline{x}, v).$$

Generalization Any smooth mapping $H: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ yields a Hamiltonian system

$$\underline{x}' = \nabla_v H(\underline{x}, v)$$

$$v' = -\nabla_{\underline{x}} H(\underline{x}, v).$$

again satisfying $\frac{d}{dt} H(\underline{x}(t), v(t)) = 0$
along solution curves.

Proof

$$\begin{aligned}\frac{d}{dt} H(\underline{x}(t), v(t)) &= \nabla_{\underline{x}} H \cdot \underline{x}' + \nabla_v H \cdot v' \\ &= \nabla_{\underline{x}} H \cdot \nabla_v H + \nabla_v H \cdot (-\nabla_{\underline{x}} H) \\ &= 0.\end{aligned}$$

Note A mechanical system is described
by $(\underline{x}, v)$. State space: $\mathbb{R}^d \times \mathbb{R}^d = \{(\underline{x}, v) \mid \underline{x}, v \in \mathbb{R}^d\}$.

Ex 2 Harmonic oscillation

$$x'' = -kx = -u'(x) \quad , \quad u(x) = \frac{kx^2}{2}$$

$$H(x, v) = \frac{v^2}{2} + u(x) \quad \text{yields}$$

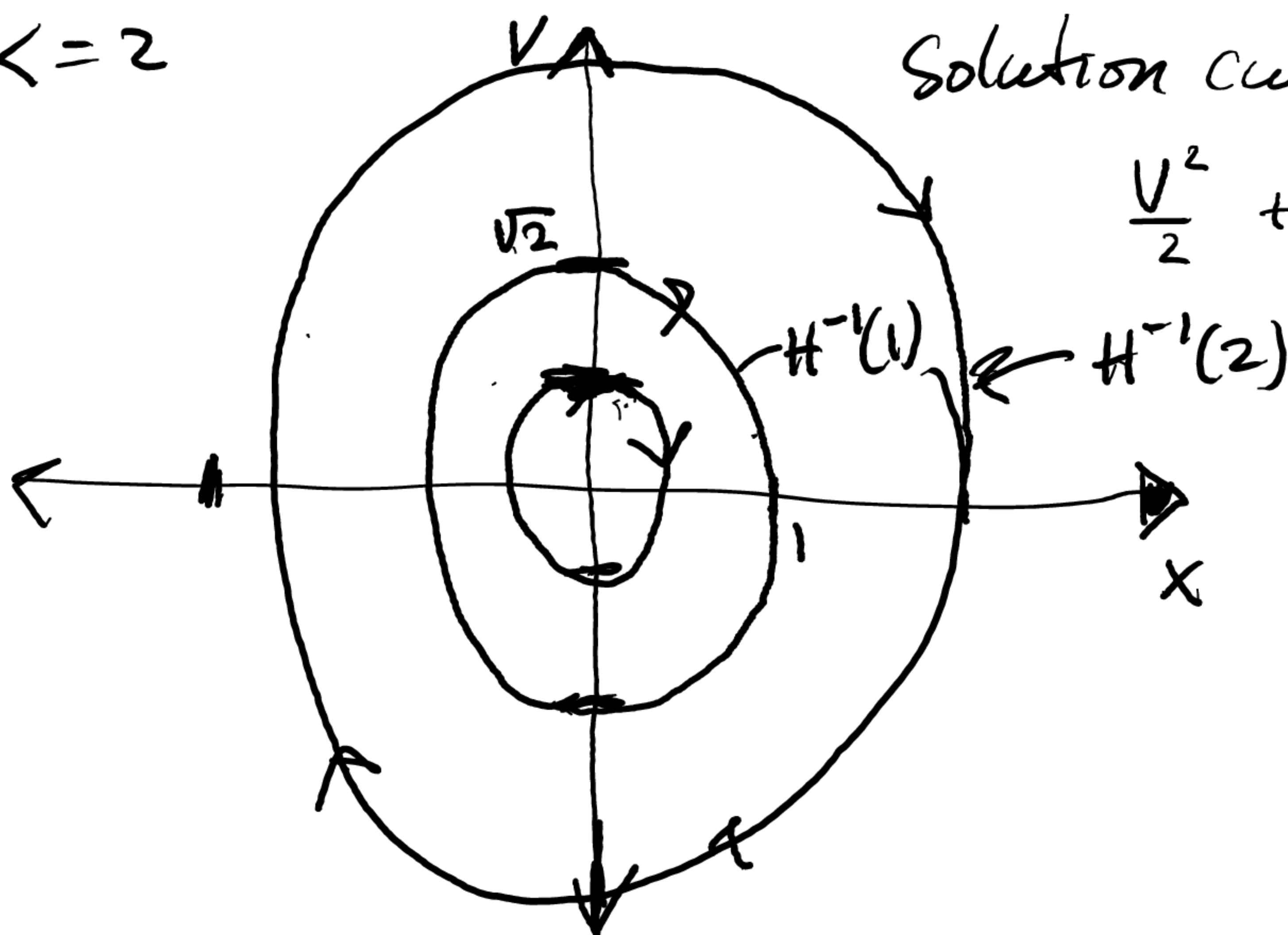
$$x' = H_v = v$$

$$v' = -H_x = -u'(x) = -kx.$$

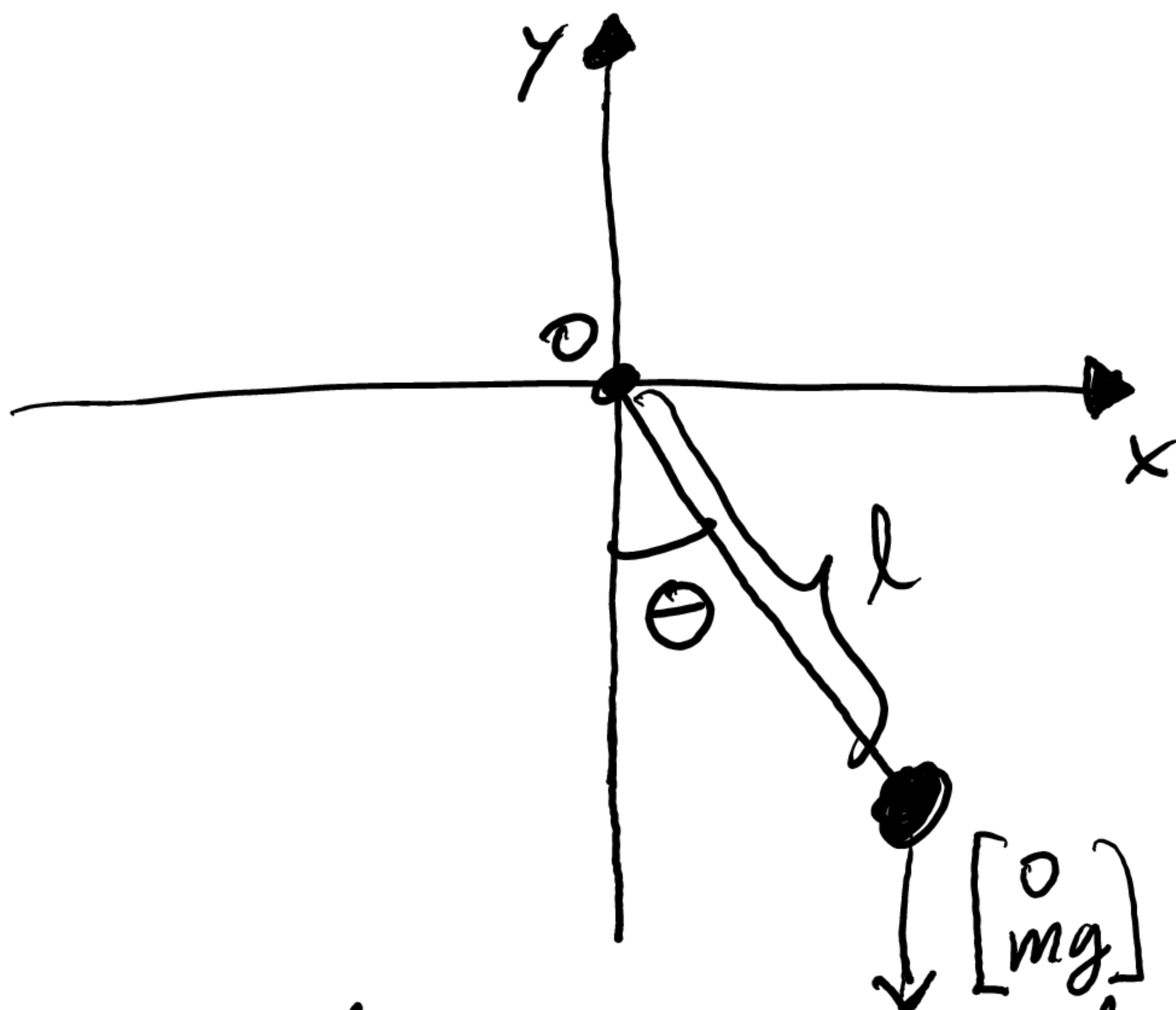
$$K=2$$

Solution curves: (x, v) st.

$$\frac{v^2}{2} + x^2 = C$$



Ex 3 Ideal Pendulum

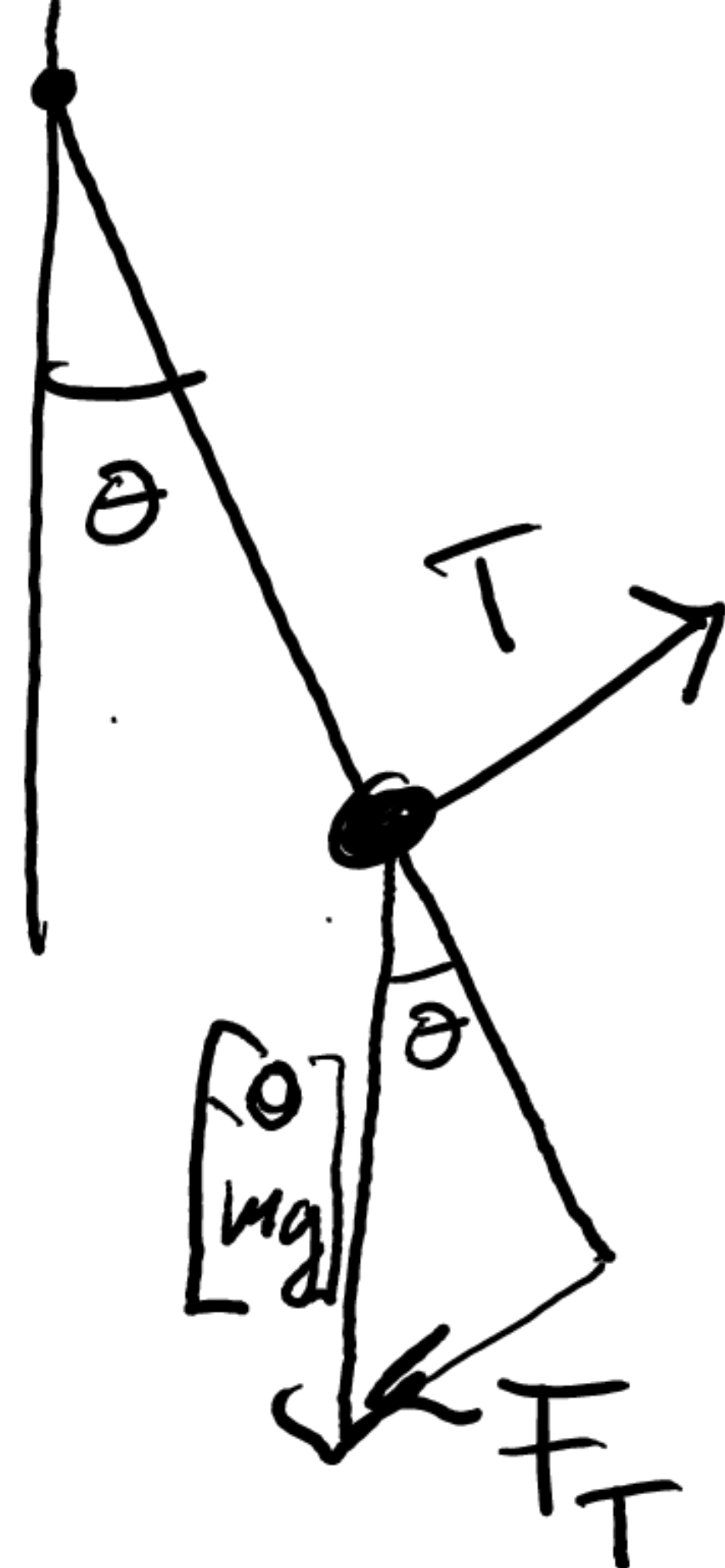


Pendulum of mass m attached to string of length l .

$$x(t) = l \sin \theta(t)$$

$$y(t) = -l \cos \theta(t)$$

$$\underline{x}' = l \theta' \underbrace{\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}}_{=:\underline{T}}$$



Tangential velocity $v := l \theta'$

$$F_T = \left(\begin{bmatrix} 0 \\ mg \end{bmatrix} \cdot \underline{T} \right) \underline{T} = -mg \sin \theta \underline{T}$$

$$F_T = m \underline{x}'' = m l (\theta')^2 \underbrace{\begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}}_{\text{Radial Force} = 0} + m l \theta'' \underline{T}$$

Radial Force = 0 $\left(\underline{T} \perp \begin{bmatrix} \sin \theta \\ -\cos \theta \end{bmatrix} \right)$

$$\Rightarrow \theta'' = -\frac{g}{l} \sin \theta$$

Equations of motion when $l = g = 1$:

$$\theta' = v$$

$$v' = \theta'' = -\sin \theta$$

Hamiltonian system

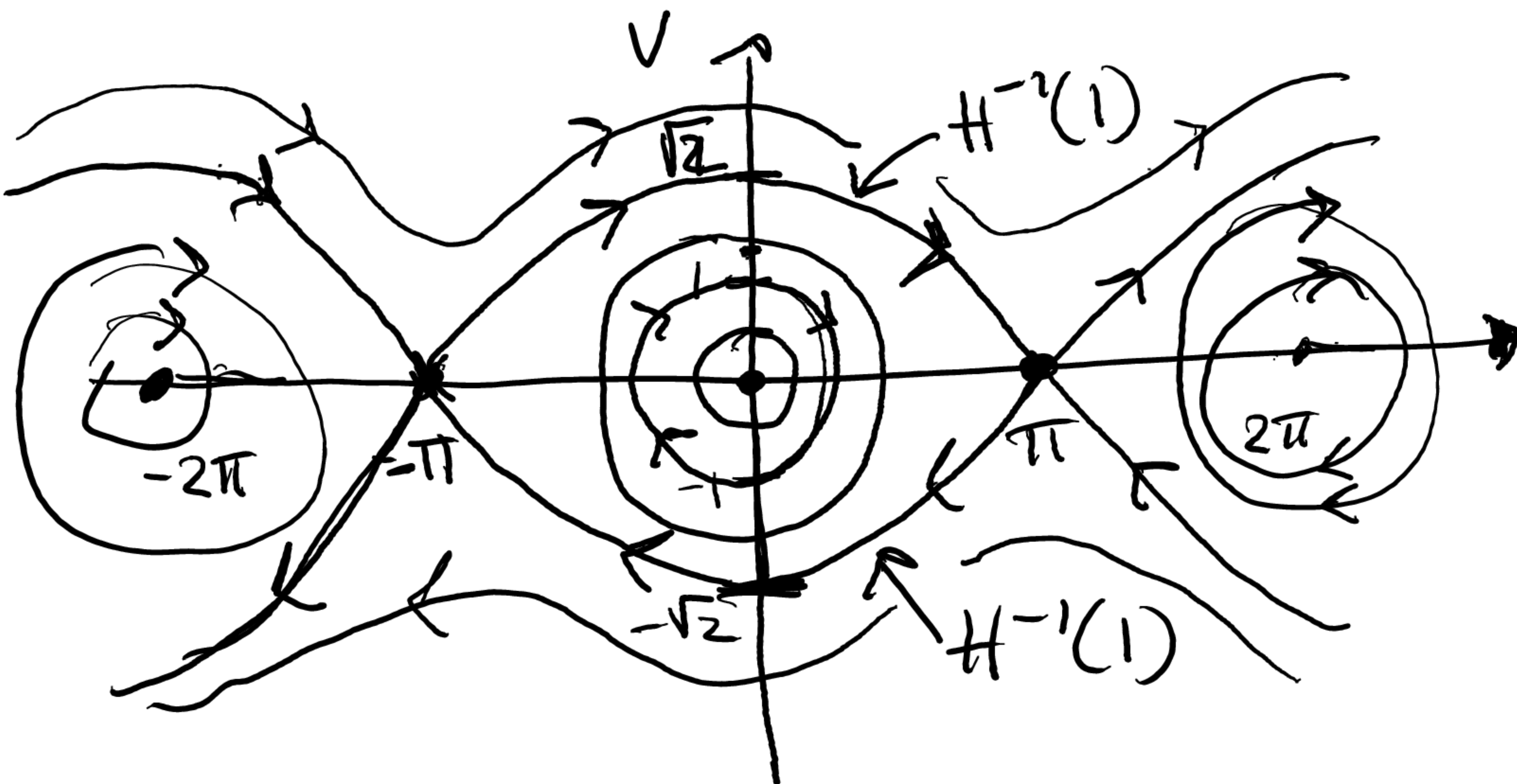
$$H(\theta, v) = \frac{v^2}{2} + -\cos \theta$$

as $\theta' = H_v$

$$v' = -H_\theta = -\sin \theta$$

Given $H(\theta(0), v(0)) = E$

$$v^2 = E + \cos \theta \quad \forall t$$



EP: $(0, 0)$, $(\pm\pi, 0)$, $(\pm 2\pi, 0)$...
stable unstable stable.

Motion around the sun

Can be well approximated by a two-body problem. By Newton's law of gravitation

$$\begin{aligned} m_1 \ddot{\mathbf{x}}_1 &= -G m_1 m_2 \frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_2 - \mathbf{x}_1|^3} \\ m_2 \ddot{\mathbf{x}}_2 &= -G m_1 m_2 \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} \end{aligned} \quad (2)$$

where $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$, $m_1 = \text{mass earth}$
 $m_2 = \text{mass sun}$.

Solve by considering

Center of mass $\mathbf{R} = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{m_1 + m_2}$

Relative motion $\mathbf{x} := \mathbf{x}_1 - \mathbf{x}_2$

with system

$$R'' = 0 \quad (R(t) = R(0) + R'(0)t)$$

$$\ddot{X} = -g(m_1 + m_2) \frac{X}{|X|^3} \quad (3)$$

The ODE for X in (3) is a central force system. Recover solution of (2)

by

$$X_1 = R + \frac{m_2}{m_1 + m_2} X$$

$$X_2 = R + \frac{m_1}{m_1 + m_2} X.$$

Central force systems Writing

$\underline{Y}(t) := X(t \sqrt{g(m_1 + m_2)})$ in (3)
yields

$$\ddot{Y} = -\frac{Y}{|Y|^3}$$

Theorem 1, A solution $(\underline{r}, \underline{r}')$ of
 (3) (a particle moving in a central force field) always moves in a fixed plane, meaning
 $\exists \underline{q} \in \mathbb{R}^3$ s.t. $\underline{q}^T \underline{r}(t) = 0 \quad \forall t \geq 0$.

Proof idea Let $\underline{r}' = \underline{v}$. Then

$$\left. \begin{aligned} \frac{d}{dt} (\underline{r} \times \underline{v}) &= \underbrace{\underline{r}' \times \underline{v}}_{=0} + \underline{r} \times \underline{v}' \\ &= \underline{r} \times \left(-\frac{\underline{r}}{|\underline{r}|^3} \right) = 0 \end{aligned} \right\} (4)$$

as $\underline{r}$ and $-\frac{\underline{r}}{|\underline{r}|^3}$ are parallel.

If $\underline{r} \times \underline{v} \neq 0$, then $\underline{q} = \underline{r} \times \underline{v}$ satisfies
 $\underline{q}^T \underline{r} = 0 \quad \forall t$.

And if $\underline{r} \times \underline{v} = 0$, then $\underline{r} \parallel \underline{v} \quad \forall t$. Hence

$$\underline{r}'(t) = g(t) \underline{r}(t) \quad \text{for some } g: \mathbb{R} \rightarrow \mathbb{R}$$

which implies

$$y_i' = g y_i \quad \text{for } i = 1, 2, 3$$

$$\Rightarrow \mathbf{Y}(t) = e^{\int_0^t g(s) ds} \mathbf{Y}(0).$$

Take any $\mathbf{q}$ s.t. $\mathbf{q} \perp \mathbf{Y}(0) \Rightarrow \mathbf{q}^T \cdot \mathbf{Y}(t) = 0$
for all $t \geq 0$ $\square$

Theorem 1 extends to any ODE on the
form $m\mathbf{Y}'' = \mathbf{F}(\mathbf{Y})$ with $\mathbf{F}(\mathbf{Y}) = \lambda(\mathbf{Y})\mathbf{Y}$.
(a central force field)

For a given $m > 0$, $m\mathbf{Y} \times \mathbf{V}$ is called
the angular momentum of an ODE
 $m\mathbf{Y}'' = \mathbf{F}(\mathbf{Y})$.

Eq (4) shows that the angular
momentum is conserved for $\mathbf{Y}'' = -\frac{\mathbf{Y}}{|\mathbf{Y}|^3}$

Writing as a Hamiltonian system

$$\begin{aligned}\frac{\partial}{\partial y_i} \frac{1}{|\mathbf{r}|} &= \frac{\partial}{\partial y_i} \frac{1}{(y_1^2 + y_2^2 + y_3^2)^{1/2}} \\ &= -\frac{1}{2} \frac{2y_i}{(y_1^2 + y_2^2 + y_3^2)^{3/2}} = -\frac{y_i}{|\mathbf{r}|^3}\end{aligned}$$

$$\Rightarrow \nabla \frac{1}{|\mathbf{r}|} = -\frac{\mathbf{r}}{|\mathbf{r}|^3}$$

Hence $\mathbf{r}'' = -\frac{\mathbf{r}}{|\mathbf{r}|^3} = -\nabla U(\mathbf{r})$

for $U(\mathbf{r}) = -|\mathbf{r}|^{-1}$. This yields

$$\mathbf{r}' = \mathbf{v} = \nabla_{\mathbf{v}} H(\mathbf{r}, \mathbf{v})$$

$$\mathbf{v}' = -\nabla U(\mathbf{r}) = -\nabla_{\mathbf{r}} H(\mathbf{r}, \mathbf{v})$$

for $H(\mathbf{r}, \mathbf{v}) = \frac{1}{2} |\mathbf{v}|^2 + U(\mathbf{r})$.

Theorem 2 Consider $\ddot{I} = -\frac{I}{|I|^3}$
and let $L = I \times V$.

If $L(0) = I(0) \times V(0) \neq 0$, then
there exists no $s > 0$ s.t. $\lim_{t \uparrow s} |I(t)| = 0$.

Proof $L \neq 0$ implies that $I(0) \neq 0$ and
 $V(0) \neq 0$ and $H(I(0), V(0)) =: E \in \mathbb{R}$.

Then $H(I(t), V(t)) = E \quad \forall t$ by Thm 1.

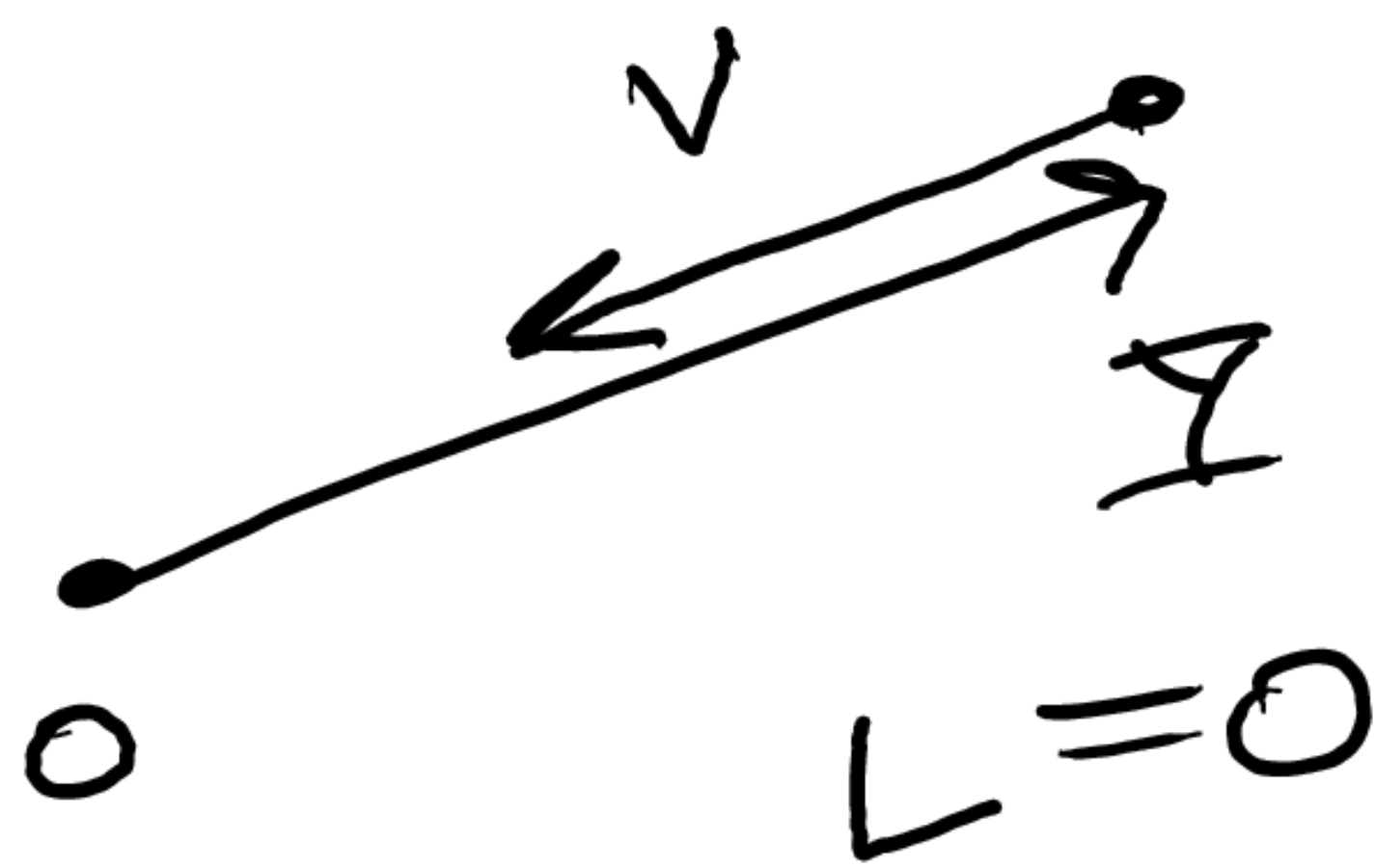
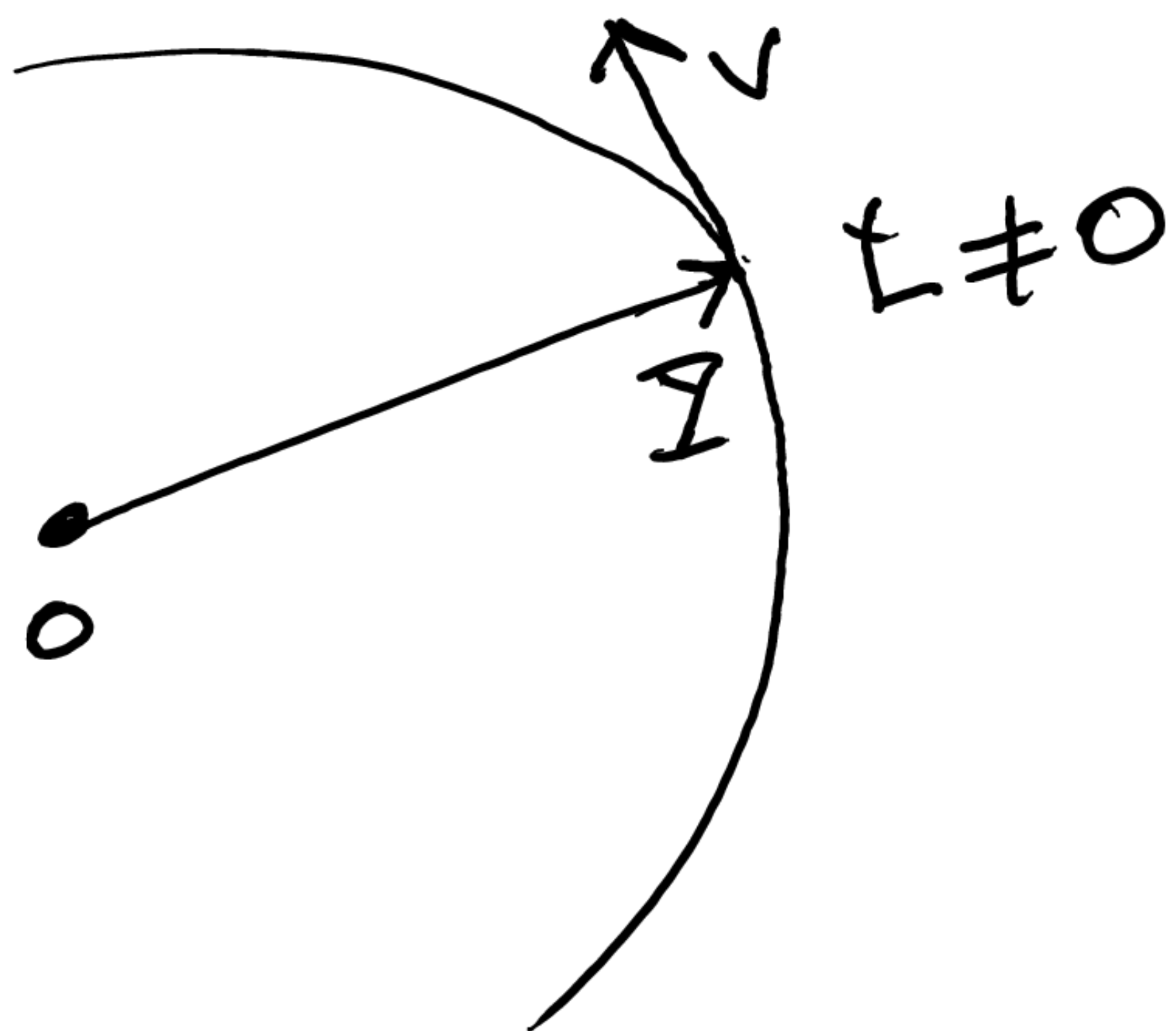
Next, $|L| = |I||V| \sin \theta \leq |I||V|$

$$\Rightarrow |V|^2 \geq \frac{|L|^2}{|I|^2} \Rightarrow H(I, V) \geq \frac{1}{|I|} \left(\frac{|L|^2}{2|I|} - 1 \right)$$

Suppose $\exists s \in (0, \infty]$ s.t. $\lim_{t \uparrow s} I(s) = 0$

This yields the contradiction

$$E = \lim_{t \uparrow s} H(I(s), V(s)) \geq \lim_{t \uparrow s} \frac{1}{|I(t)|} \left(\frac{|L|^2}{2|I|} - 1 \right) = \infty.$$



and particle
will hit $\odot$
eventually!