

lecture 28

A nonlinear ODE

$$\dot{x} = F(x) \quad \text{with } F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$$

has a positively invariant region $D \subset \mathbb{R}^n$
if $x_0 \in D \Rightarrow \phi_t(x_0)$ is defined
and in D for all $t \geq 0$.

Shorthand: $\phi_t(D) \subset D \quad \forall t \geq 0$.

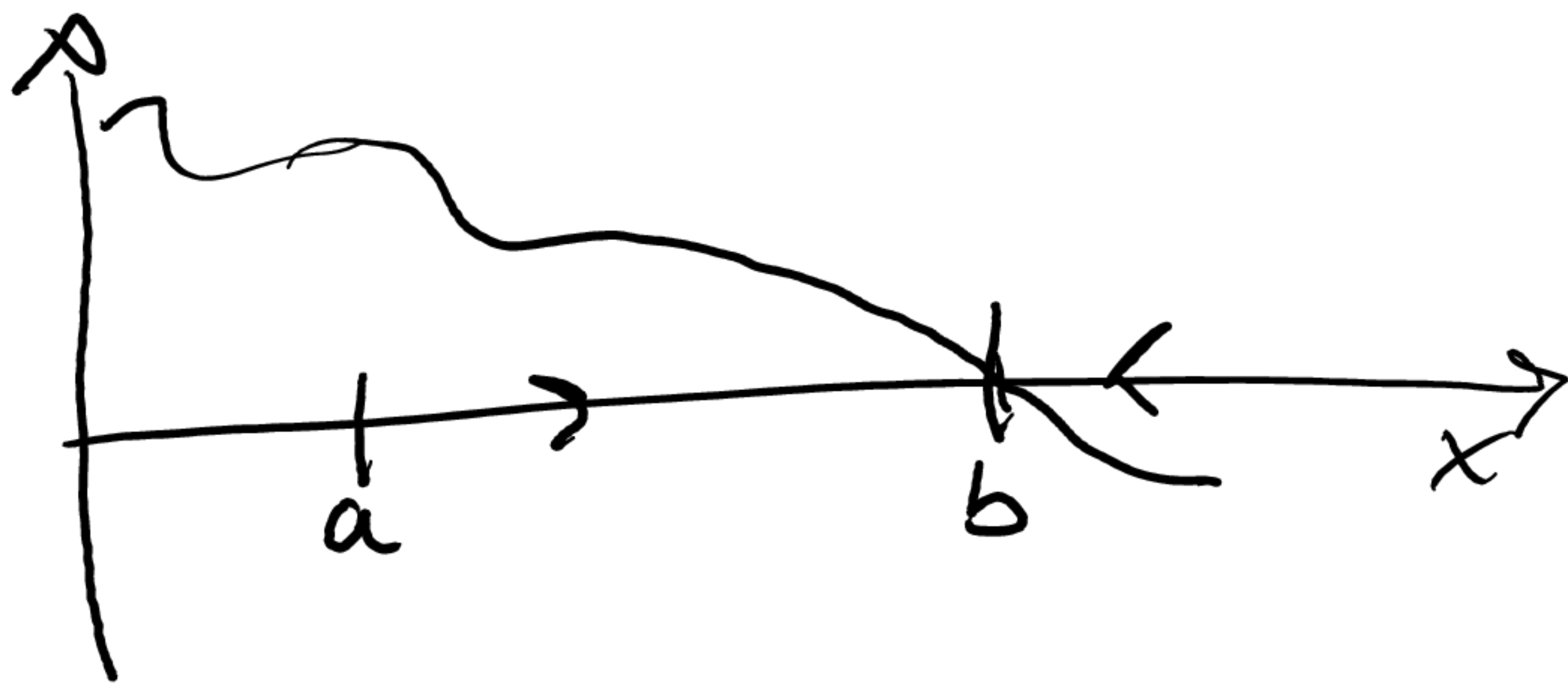
Consequence: If D is a bounded
positively invariant region and
 $x_0 \in D$, then

$$|x(t)| = |\phi_t(x_0)| \leq \sup_{z \in D} |z| \quad \forall t \geq 0.$$

Q: Sufficient conditions to have
an invariant region?

$$n=1$$

$$x' = f(x)$$

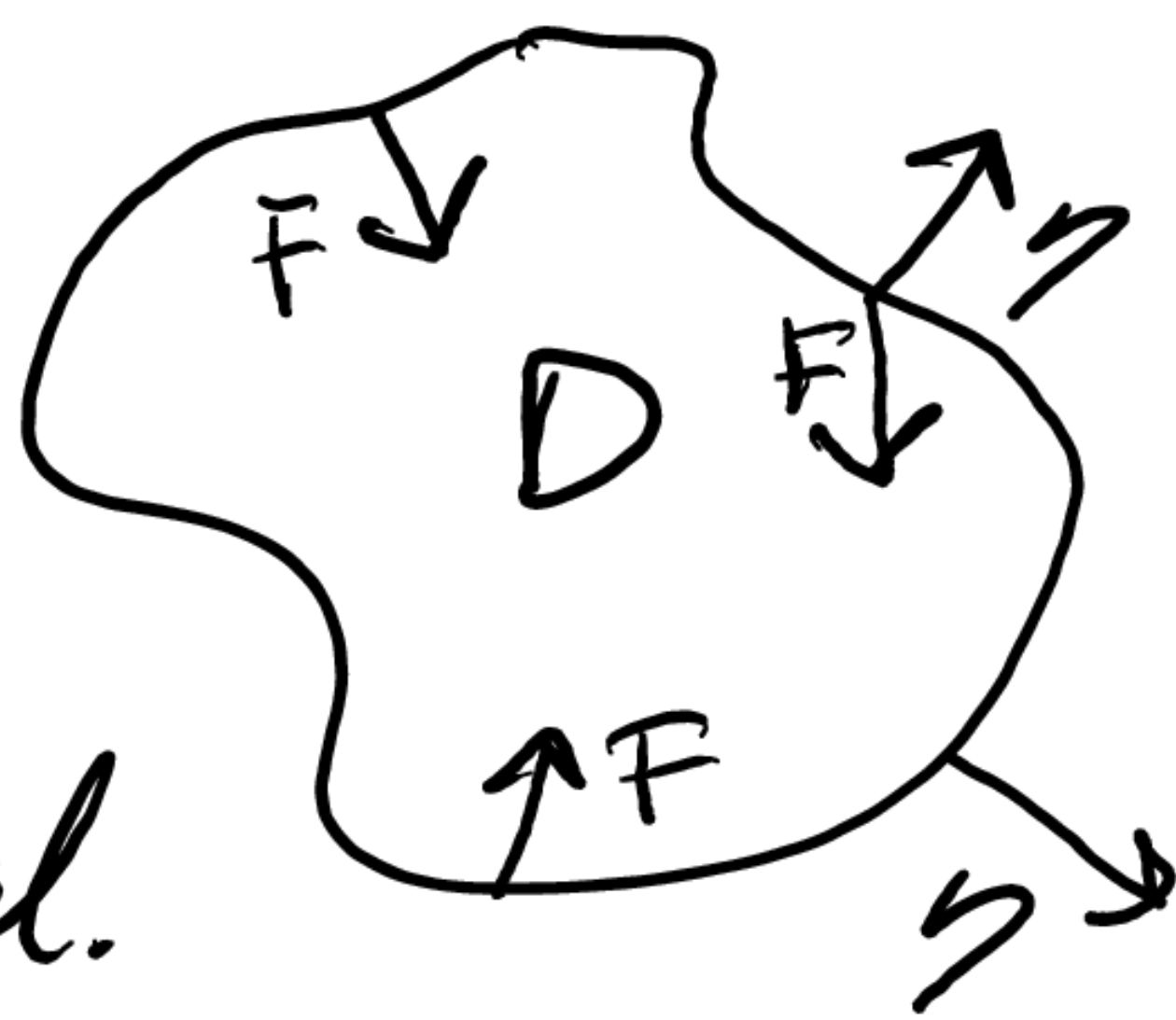


$D = [a, b]$ where b is stable EP

sufficient: $f(a) \geq 0, f(b) \leq 0$.

$d=n$ Let $D \subset \mathbb{R}^2$ be closed, connected and bdd with ∂D smooth and $\eta: \partial D \rightarrow \mathbb{R}^2$

the outward pointing normal.



Then D is positively invariant

if $F(x) \cdot \eta(x) \leq 0 \quad \forall x \in \partial D$

(meaning F does not point out of D on ∂D)

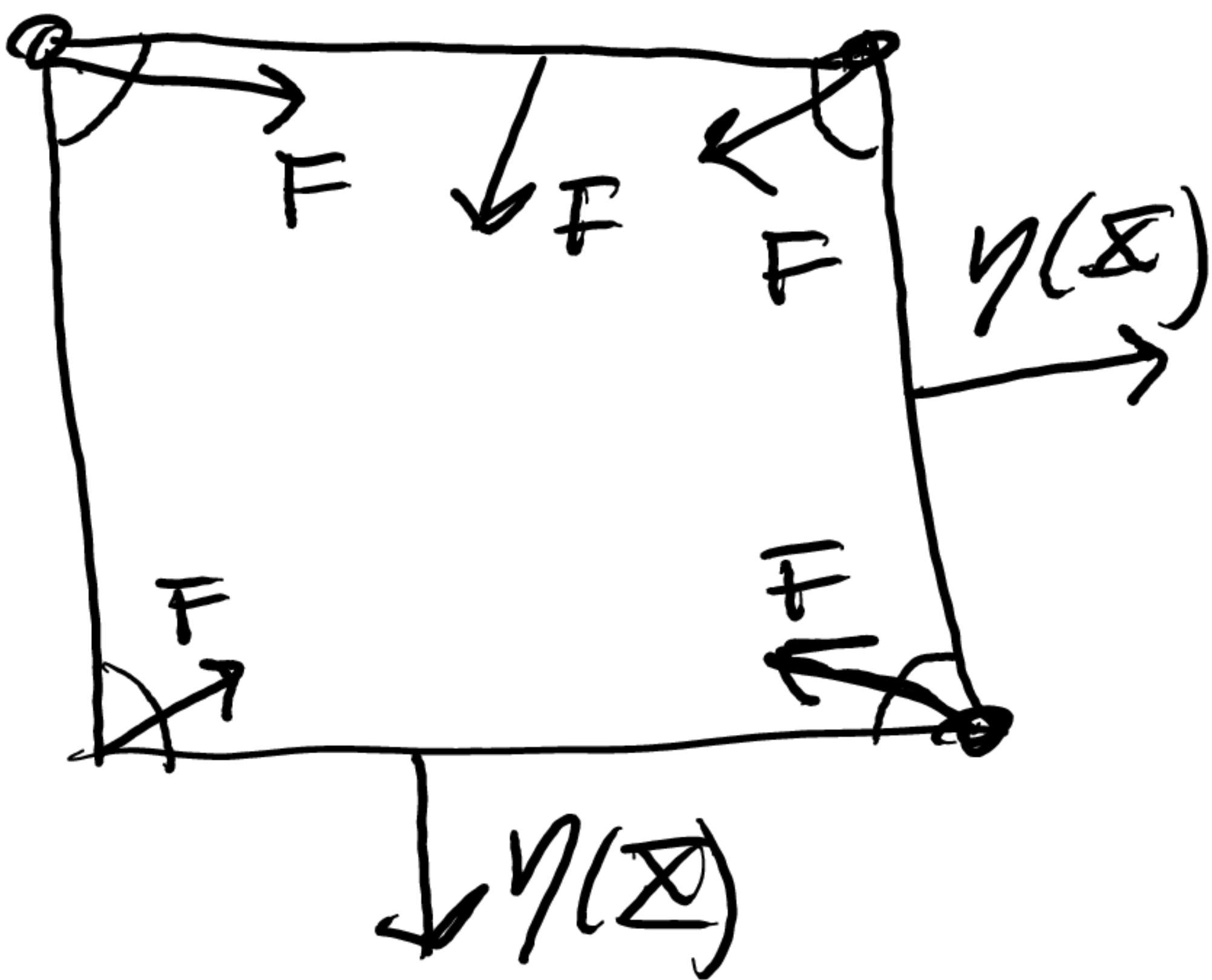
This extends to regions with corners

For example, for $D = [0, 1] \times [0, 1]$,

$F(x) \cdot \eta(x) \leq 0$ for $x \in \partial D$ not a
corner point

and $f_1(0,0) \geq 0, f_2(0,0) \geq 0$

$f_1(1,0) \leq 0, f_2(1,0) \geq 0$ etc.



Nullclines

For $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{bmatrix}$

the set

$$\{ \mathbf{x} \mid f_i(\mathbf{x}) = 0 \} \text{ is}$$

called the x_i -nullcline. Will split nullclines into subsets and also call subsets x_i -nullclines.

Ex 1 Lotka-Volterra $n=2$
(competitive species)

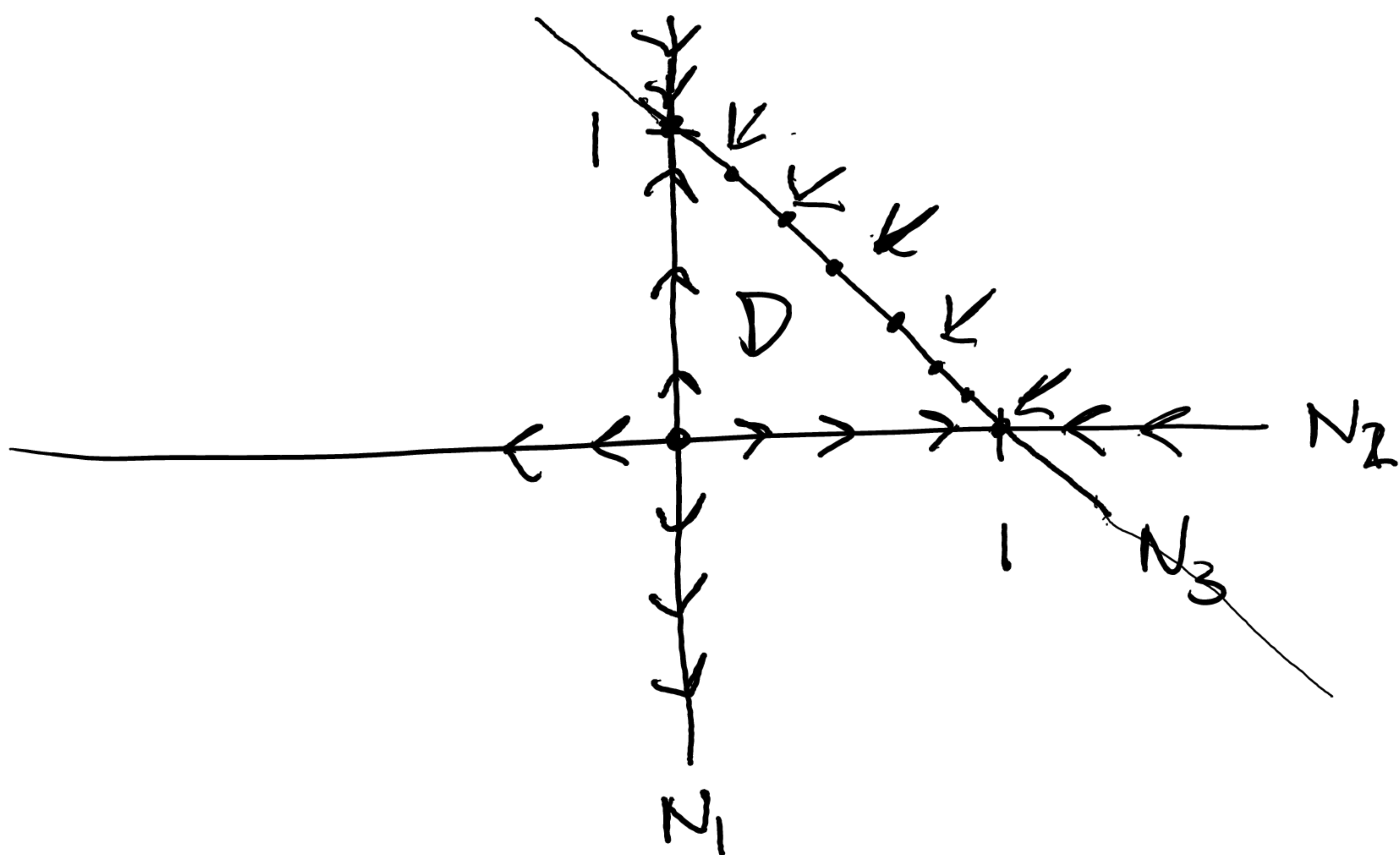
$$\dot{x}_1 = x_1(1 - x_1 - x_2)$$

$$\dot{x}_2 = x_2(1 - x_1 - x_2)$$

x_1 -nullcline $x_1 = 0$ N_1

x_2 -nullcline $x_2 = 0$ N_2

x_1 & x_2 -nullcline $x_1 + x_2 = 1$ N_3



$$F|_{N_1} = \begin{bmatrix} 0 \\ x_2(1-x_2) \end{bmatrix}, \quad F|_{N_2} = \begin{bmatrix} x_1(1-x_1) \\ 0 \end{bmatrix}$$

$(0,0), (1,0), (0,1)$ are EP

$$F|_{N_3} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{as } N_3 \ni x_1 \text{ \& } x_2 - \text{nullcline}$$

$D = \{(x_1, x_2) \mid x_1, x_2 \geq 0 \text{ \& } x_1 + x_2 \leq 1\}$
is a positively invariant set

as F does not point outward on ∂D .

Applications in biology

SIR model

$$S' = -\beta SI$$

susceptible

$$I' = \beta SI - \nu I$$

infected

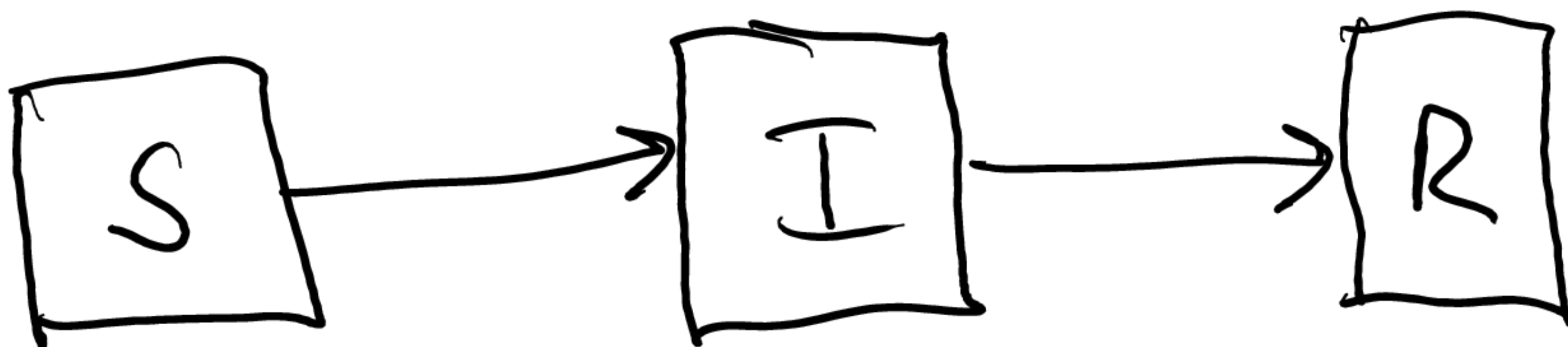
$$R' = \nu I$$

recovered

with $\beta, \nu > 0$.

Properties

- $N := S + I + R$ is constant $N' = 0$.
- Seek solutions where $S, I, R \geq 0 \forall t \geq 0$.
- Is a compartment model



Note Suffices to consider closed system

$$S' = -\beta SI \quad (1)$$

$$I' = \beta SI - \nu I = (\beta S - \nu) I$$

as $R = N - (S + I)$ (and $N = S_0 + I_0 + R_0$).

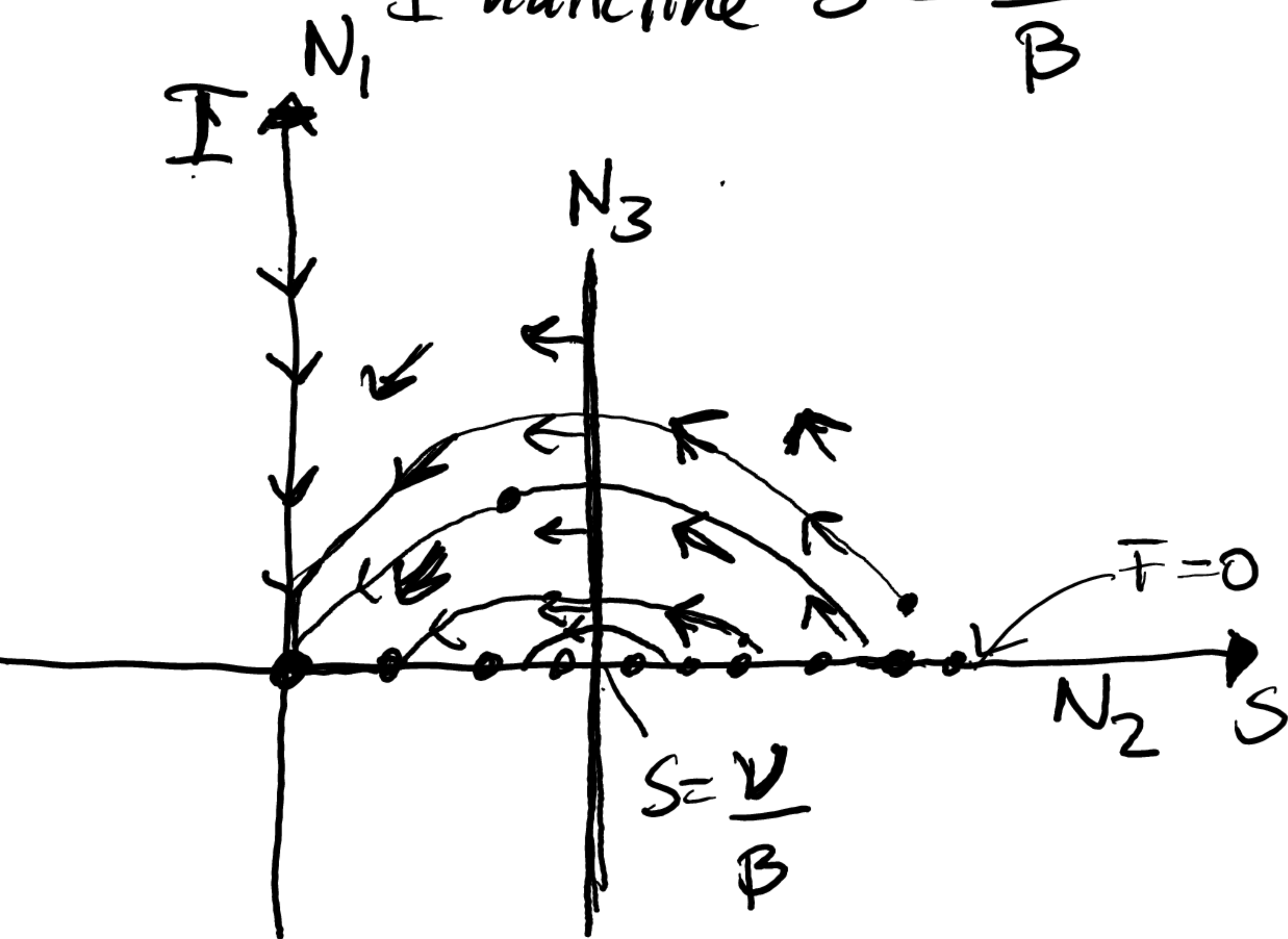
Seek to verify that if $S_0, I_0 \geq 0$,
then $S, I \geq 0 \quad \forall t \geq 0$.

Nullclines of (1):

S-nullcline $S = 0$ N_1

S&I-nullcline $I = 0$ N_2

I-nullcline $S = \frac{\nu}{\beta}$ N_3



$$F|_{N_1} = \begin{bmatrix} 0 \\ -\nu I \end{bmatrix}$$

$$F|_{N_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$F|_{N_3} = \begin{bmatrix} -\nu I \\ 0 \end{bmatrix}$$

Equilibria

$F=0$ on N_2

Any solution with $S_0, I_0 \geq 0$ can only become negative by crossing N_1 leftward or N_2 downward. This is not possible as

$$F|_{N_1} = \downarrow \text{ and } F|_{N_2} = 0.$$

$$(F|_{N_1} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} \leq 0 \text{ and } F|_{N_2} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} \leq 0).$$

$\Rightarrow S, I \geq 0$ for all times when the solution exists.

$$\text{Further, } (S+I)' = -\nu I \leq 0$$

$$\Rightarrow S+I \leq S_0+I_0 \quad \forall t \geq 0$$

so a unique solution exists for all times and

$D = \{ (S, I) \mid S, I \geq 0 \}$ is positively invariant.

Supposing $I_0, S_0 > 0$, we further have for all t s.t. $S, I > 0$ that

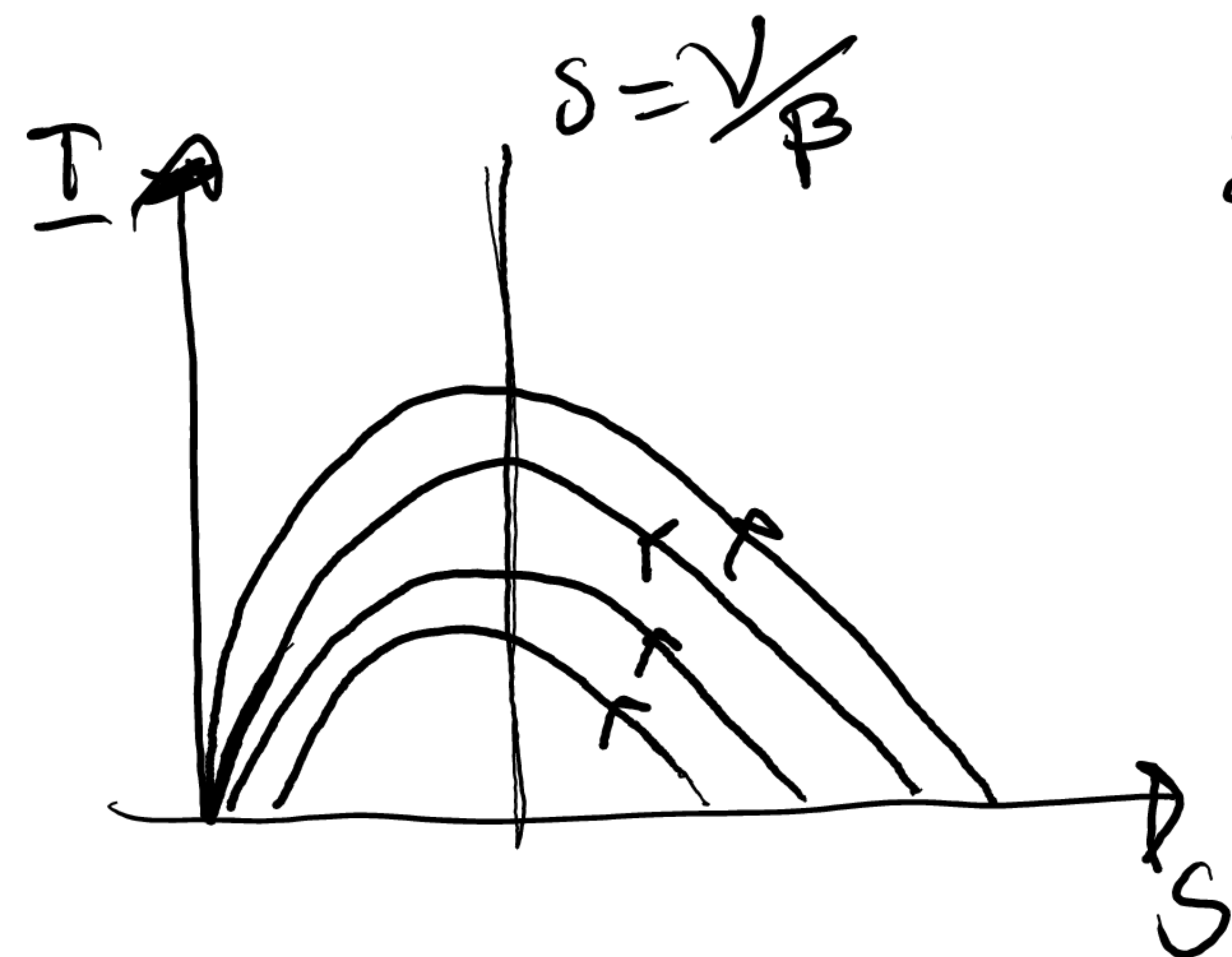
$$\frac{I'}{S'} = \frac{\beta SI - \nu I}{-\beta SI} = -1 + \frac{\nu}{\beta S}$$

$S(t)$ strictly monotone $\Rightarrow S \mapsto t(S)$ invertible

and $\frac{dI}{dS} = \frac{dI/dt}{dS/dt} = -1 + \frac{\nu}{\beta S}$

$$\Rightarrow I(S) = -S + \frac{\nu}{\beta} \log S + C$$

Solution curves: $I + S - \frac{\nu}{\beta} \log S = C$.



$S = \frac{\nu}{\beta}$ is a threshold level for infection growth:

$$I'(S) > 0 \quad S > \frac{\nu}{\beta}$$

$$I'(S) < 0 \quad S < \frac{\nu}{\beta}$$

Lotka-Volterra equation (Predator-Prey)

$$x' = x(a - by)$$

$$y' = y(-c + dx)$$

where $a, b, c, d > 0$.

Prey $x' = \underbrace{ax}_{\text{current population}} - \underbrace{bxy}_{\propto \# \text{Predator Prey meetings}}$

Predator $y' = \underbrace{-cy}_{\propto \text{current population}} + \underbrace{dxy}_{\propto \# \text{predator prey meetings}}$

EP: $(0, 0), \left(\frac{c}{d}, \frac{a}{b}\right)$

$DF(x) = \begin{bmatrix} a - by & -bx \\ dy & -c + dx \end{bmatrix}, \quad DF(0) = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$

$\Rightarrow 0$ is a saddle.

$$DF\left(\begin{bmatrix} c/d \\ a/b \end{bmatrix}\right) = \begin{bmatrix} 0 & -\frac{bc}{d} \\ \frac{da}{b} & 0 \end{bmatrix}$$

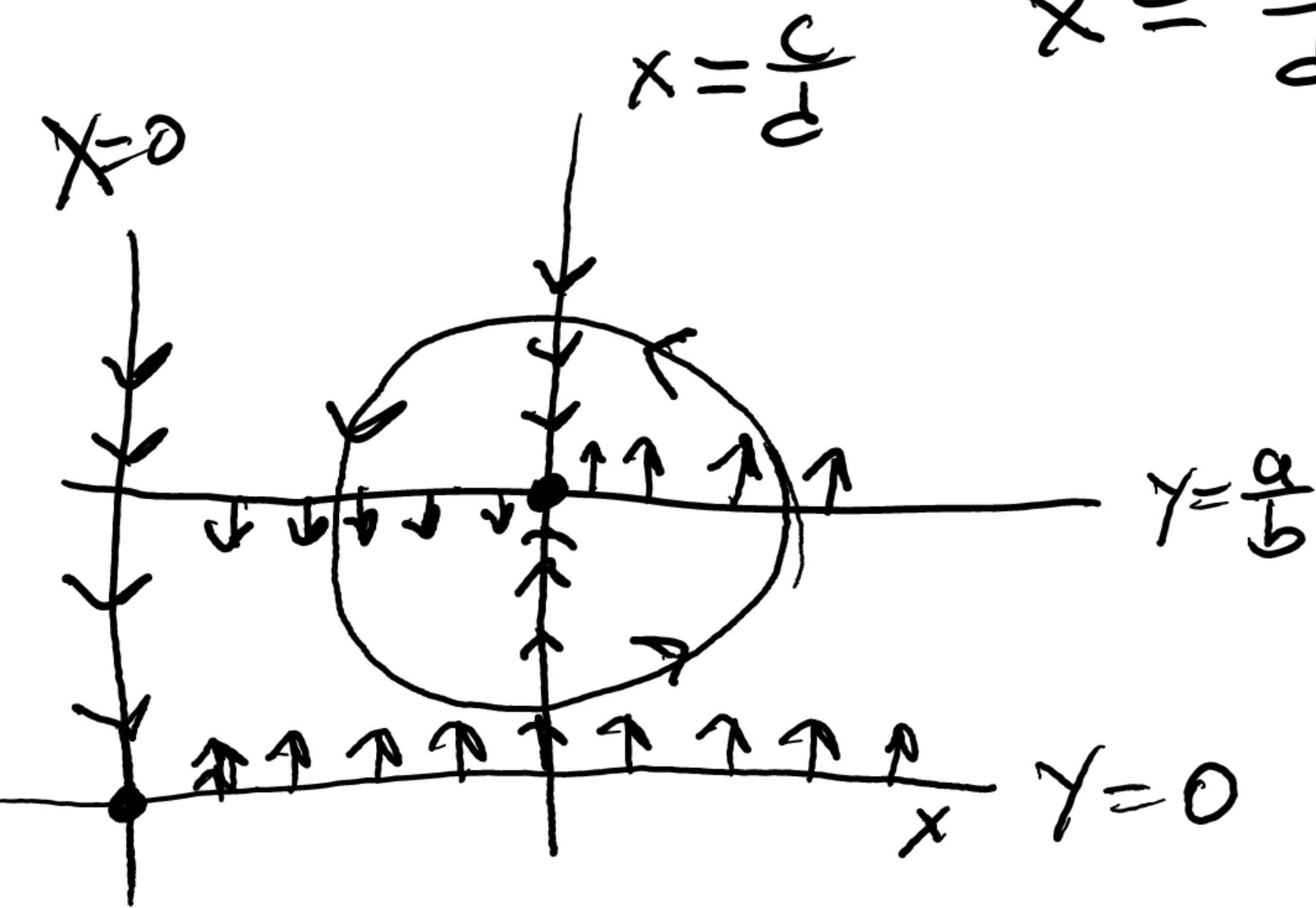
has $\lambda_{\pm} = \pm i\sqrt{ac}$, so matrix is not hyperbolic and cannot classify EP by linearization.

Nullclines

x-nullclines $x=0 : N_1$, $F|_{N_1} = \begin{bmatrix} 0 \\ -cy \end{bmatrix}$
 $y = \frac{a}{b} : N_2$, $F|_{N_2} = \begin{bmatrix} 0 \\ \frac{a}{b}(-c+dx) \end{bmatrix}$

y-nullclines $y=0 : N_3$, $F|_{N_3} = \begin{bmatrix} ax \\ 0 \end{bmatrix}$
 $x = \frac{c}{d} : N_4$,

$$F|_{N_4} = \begin{bmatrix} \frac{c}{d}(a-by) \\ 0 \end{bmatrix}$$



Nullclines show that if

$x_0, y_0 \geq 0$, then $x, y > 0$ as long as solution exists, since

$$F|_{N_1} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix} \leq 0 \quad \& \quad F|_{N_3} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} \leq 0.$$

Q: Are solutions closed orbits, or do they spiral into EP or outward?

Derivation of Lyapunov fcn for $\bar{x}^* = \begin{bmatrix} c/d \\ a/b \end{bmatrix}$

Suppose $L(\bar{x}) = F(x) + G(y)$.

$$\begin{aligned} \text{Then } \frac{d}{dt} L(\bar{x}(t)) &= F'(x) x' + G'(y) y' \\ &= x F'(x)(a - by) - y G'(y)(c - dx) \\ &=: \dot{L}(\bar{x}) \end{aligned}$$

If we can find F, G s.t.

$$\frac{x F'(x)}{dx - c} = \frac{y G'(y)}{by - a} \stackrel{x, y \text{ independent variables.}}{=} \hat{C} \text{ constant,}$$

then the goal $\dot{L}(x, y) = 0$ holds.

Solving for $\hat{C} = 1$:

$$F'(x) = d - \frac{c}{x}$$

$$G'(y) = b - \frac{a}{y}$$

yields a solution

$$F(x) = dx - c \log x$$

$$G(y) = by - a \log y$$

The function

$$L(x, y) = F(x) + G(y)$$

$$= dx + by - c \log x - a \log y$$

is constant along solution paths,

$\nabla L(\bar{x}^*) = 0$ & L is strictly convex
on $(0, \infty)^2 \Rightarrow \bar{x}^*$ is a strict minimum,

and $\dot{L}(\bar{x}) \leq 0$ on $(0, \infty)^2$

$\Rightarrow \tilde{L}(\bar{x}) = L(\bar{x}) - L(\bar{x}^*)$ is
a Lyapunov fcn for $\bar{x}^*$ on $(0, \infty)^2$.

Hence $\bar{x}^*$ is a stable EP.

And given $x_0, y_0 > 0$,

$$L(\bar{x}(t)) = L(x_0, y_0)$$

implies that $x(t)$ and $y(t)$
are bounded for all times.

(As $\lim_{x \rightarrow \infty} L(x, y) = \infty \quad \forall y > 0$
& $\lim_{y \rightarrow \infty} L(x, y) = \infty \quad \forall x > 0$
implies $L^{-1}(L(x_0, y_0))$ is bounded)

So $D = (0, \infty)^2$ is a positively invariant region.

One can further show that for any $(x_0, y_0) \in (0, \infty)^2 \setminus \Sigma^*$

$L^{-1}(L(x_0, y_0)) = \{ (x, y) \mid L(x, y) = L(x_0, y_0) \}$
 is a simple closed curve (forms a loop and does not cross itself)

Hence, for any $(x_0, y_0) \in (0, \infty)^2 \setminus \Sigma^*$,
 the unique solution $\phi_t(x_0, y_0)$
 is periodic with $\bigcup_{t \geq 0} \phi_t(x_0, y_0) = L^{-1}(L(x_0, y_0))$

