

Lecture 29

Periodic solutions of ODE in 2D

Consider $\underline{x}' = F(\underline{x})$.

If there for some $\underline{x} \in \mathbb{R}^2$ and

$\tau > 0$ holds that $\phi_\tau(\underline{x}) = \underline{x}$,

then $\{\phi_t(\underline{x})\}_{t \geq 0}$ is a periodic solution

closed orbit and

τ is the period of the solution.



A closed orbit $\gamma \subset \mathbb{R}^2$ is always bounded (it is also called a simple closed curve).

Ex 1

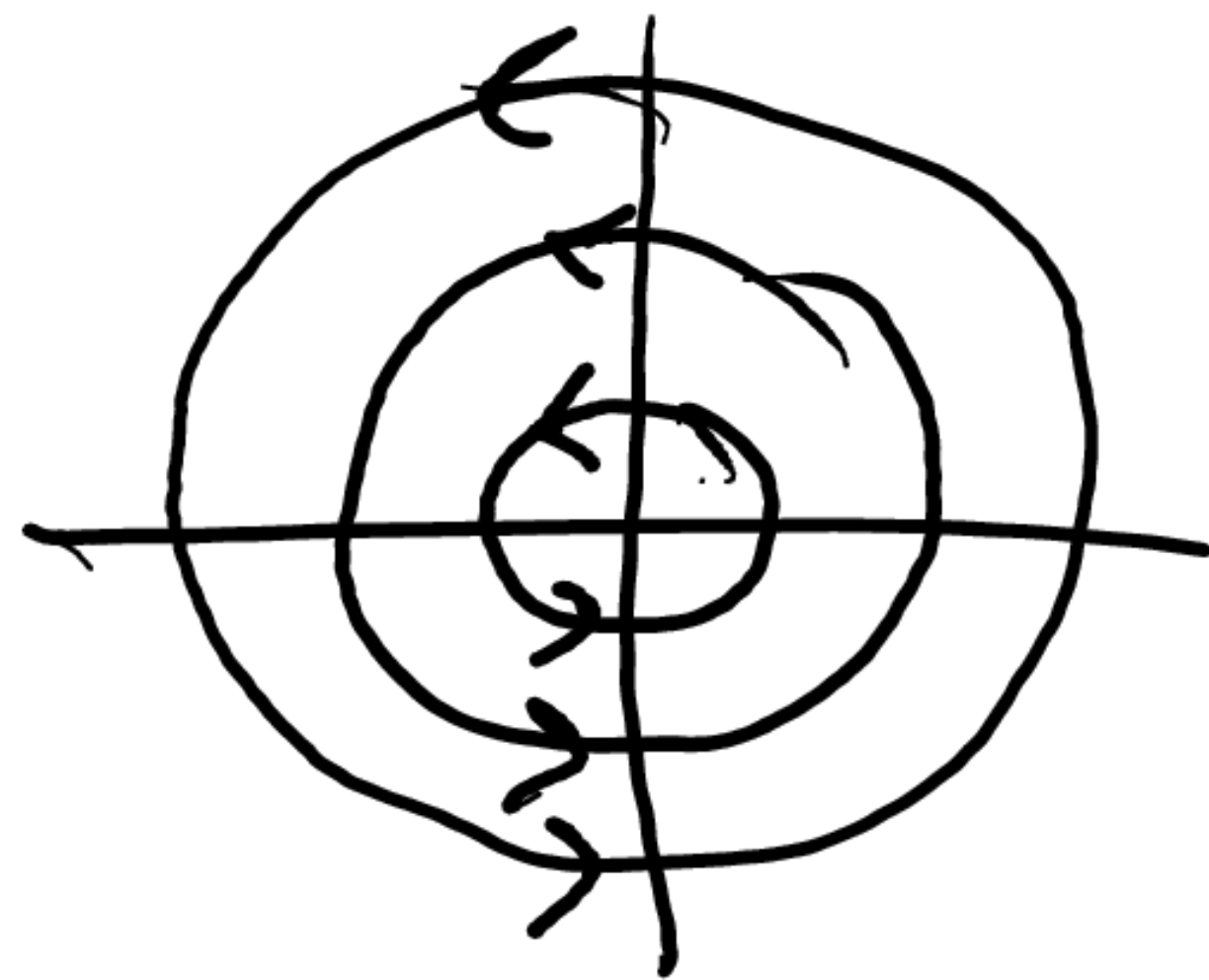
$$\left. \begin{array}{l} x' = y \\ y' = -x \end{array} \right\} \text{rewrite} \quad \begin{array}{l} r' = 0 \\ \theta' = 1 \end{array}$$

has periodic solutions

$$x(t) = r \cos(t + \theta_0) \quad \text{for each } \mathbb{Z} \in \mathbb{R}^2 \setminus \{0\}$$

$$y(t) = r \sin(t + \theta_0)$$

all with period $T = 2\pi$.



Recall that $D \subset \mathbb{R}^2$ is positively
invariant if $\phi_t(D) \subset D \quad \forall t \geq 0$

and the limit sets

$$\bullet \omega(\mathbb{Z}) = \left\{ \mathbb{Z} \mid \exists \{t_n\}_{n=1}^{\infty} \subset (0, \infty) \text{ with } t_n \rightarrow \infty \right. \\ \left. \text{s.t. } \lim_{n \rightarrow \infty} \phi_{t_n}(\mathbb{Z}) = \mathbb{Z} \right\}$$

$$\bullet \alpha(\mathbb{Z}) = \left\{ \mathbb{Z} \mid \exists \{t_n\}_{n=1}^{\infty} \subset (-\infty, 0) \text{ with } t_n \rightarrow -\infty \right. \\ \left. \text{s.t. } \lim_{n \rightarrow \infty} \phi_{t_n}(\mathbb{Z}) = \mathbb{Z} \right\}$$

Theorem 1 Poincaré-Bendixson

Let $\mathbf{X}' = F(\mathbf{X})$ with $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$.
If a non-empty set $D \subset \mathbb{R}^2$ is compact and positively invariant, and D does not contain any EP, then $W(\mathbf{X}) \subset D$ is a closed orbit for each $\mathbf{X} \in D$. (So then D contains at least one periodic solution.)

NB! Any compact D that is simply connected (no holes) contains at least one EP!

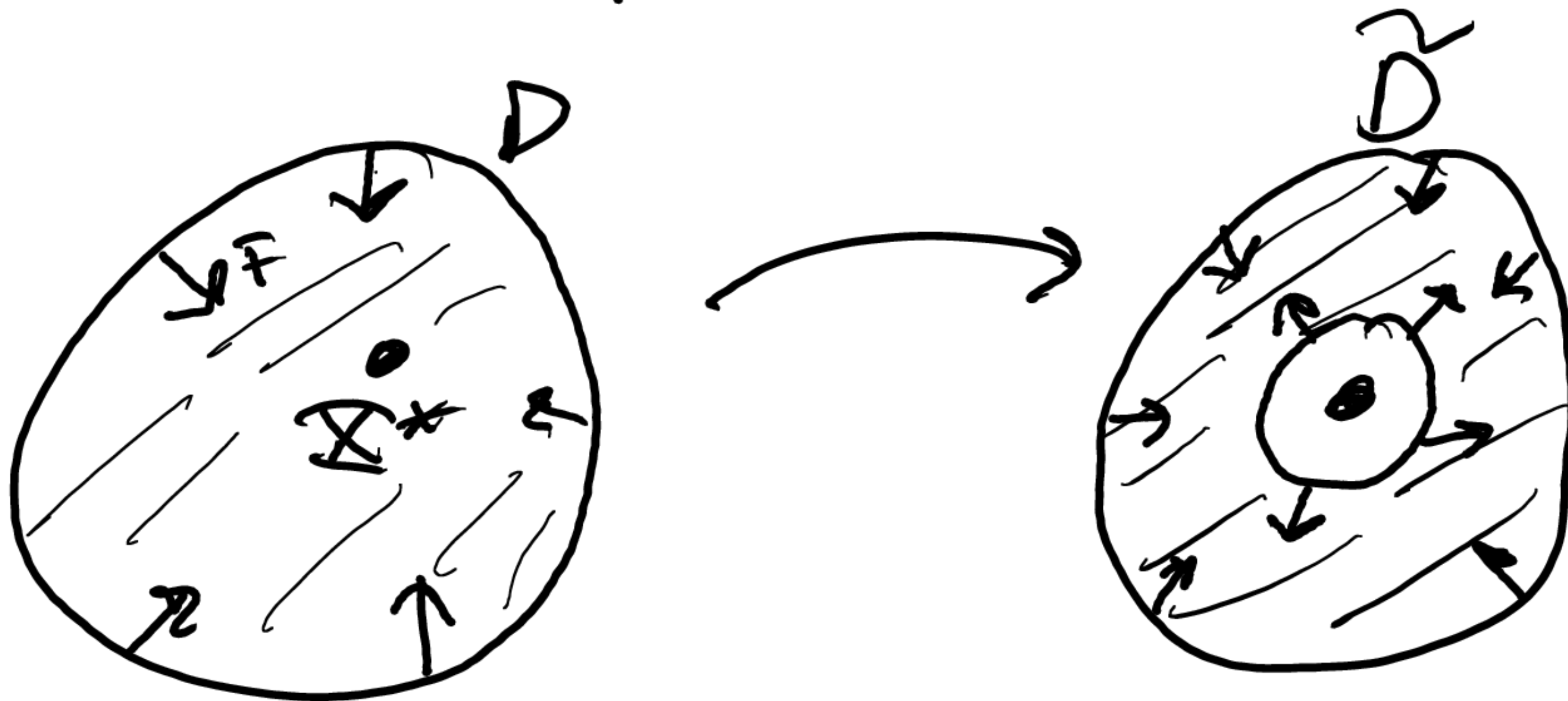
$\Rightarrow D$ in Thm 1 must have at least one hole (to have no EP).

Typical use of Thm 1:

1. Find invariant region $D \subset \mathbb{R}^2$
2. For each EP $\mathbf{X}^*$ in D cut out neighborhood A of $\mathbf{X}^*$ from D

$$\tilde{D} = D \setminus A$$

3. Check if $\tilde{D}$ is positively invariant



Ex 2 Consider

$$r' = -r(r-1)(r-2)(r-3) = f_1(r)$$

$$\theta' = 1 = f_2(r, \theta)$$

on $D = \{ (r, \theta) \mid 0.1 \leq r \leq 3 \}$



shows that if $r_0 \in D$, then

$$r(t) \in [0.1, 3] \quad \forall t \geq 0$$

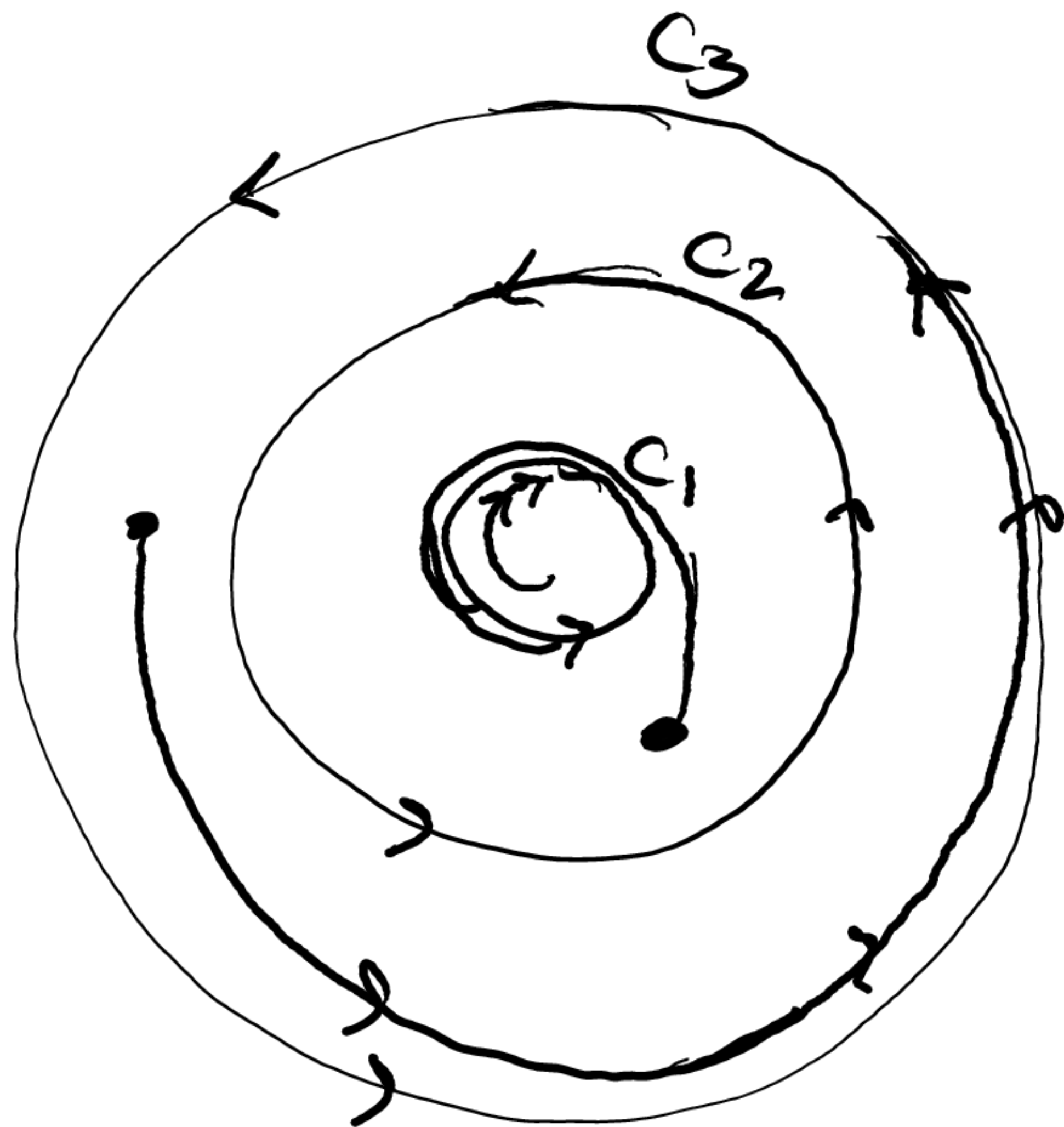
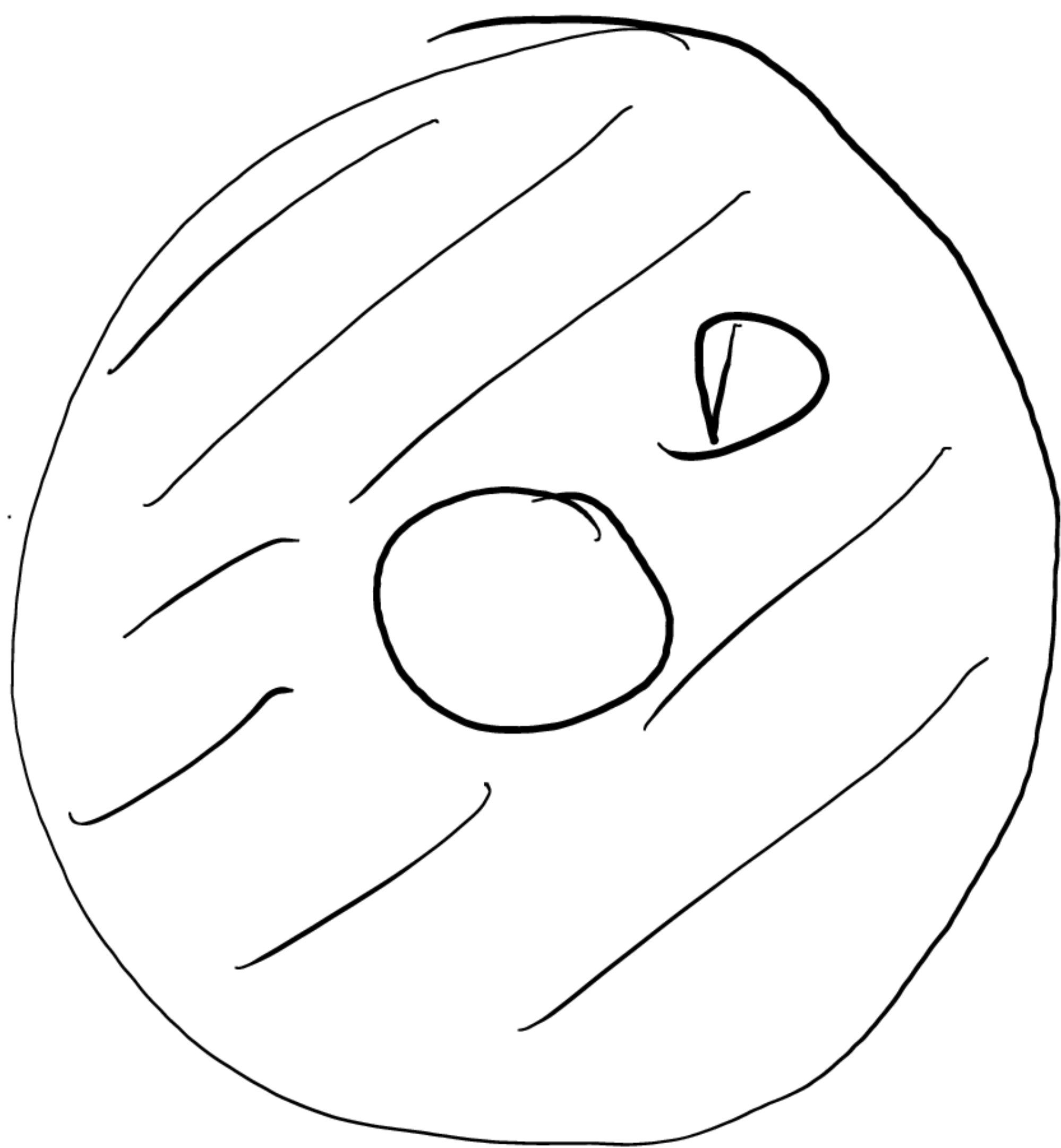
$\Rightarrow D$ is positively invariant.

closed orbits:

$$C_i = \partial B_i(0) \quad i=1,2,3$$

and

$$\omega(Z_0) = \begin{cases} C_1 & \text{if } |Z_0| \in (0, 2) \\ C_2 & \text{if } |Z_0| = 2 \\ C_3 & \text{if } |Z_0| > 2. \end{cases}$$



Note Theorem 1 does not hold
for $D = \overline{B}(0, 3)$ as for
 $\bar{x} = 0$, $w(0) = 0$ (not a closed
orbit).

Def A limit cycle $\gamma \subset \mathbb{R}^2$ is
a closed orbit s.t. $\gamma \subset \omega(\mathbb{X})$
or $\gamma \subset \alpha(\mathbb{X})$ for some $\mathbb{X} \notin \gamma$.

• For

$$r' = r(1-r^2)$$

$$\theta' = 1$$

$\gamma = \partial B_1(0) \subset \omega([1])$ is an ω -limit
cycle.

• The ODE
$$\left. \begin{array}{l} x' = y \\ y' = -x \end{array} \right\} \quad \begin{array}{l} r' = 0 \\ \theta' = 1 \end{array}$$

has no limit cycles.

(as $\mathbb{X} \in \omega(\mathbb{X}) = \partial B_{|\mathbb{X}|}(0) \quad \forall |\mathbb{X}| \neq 0$).

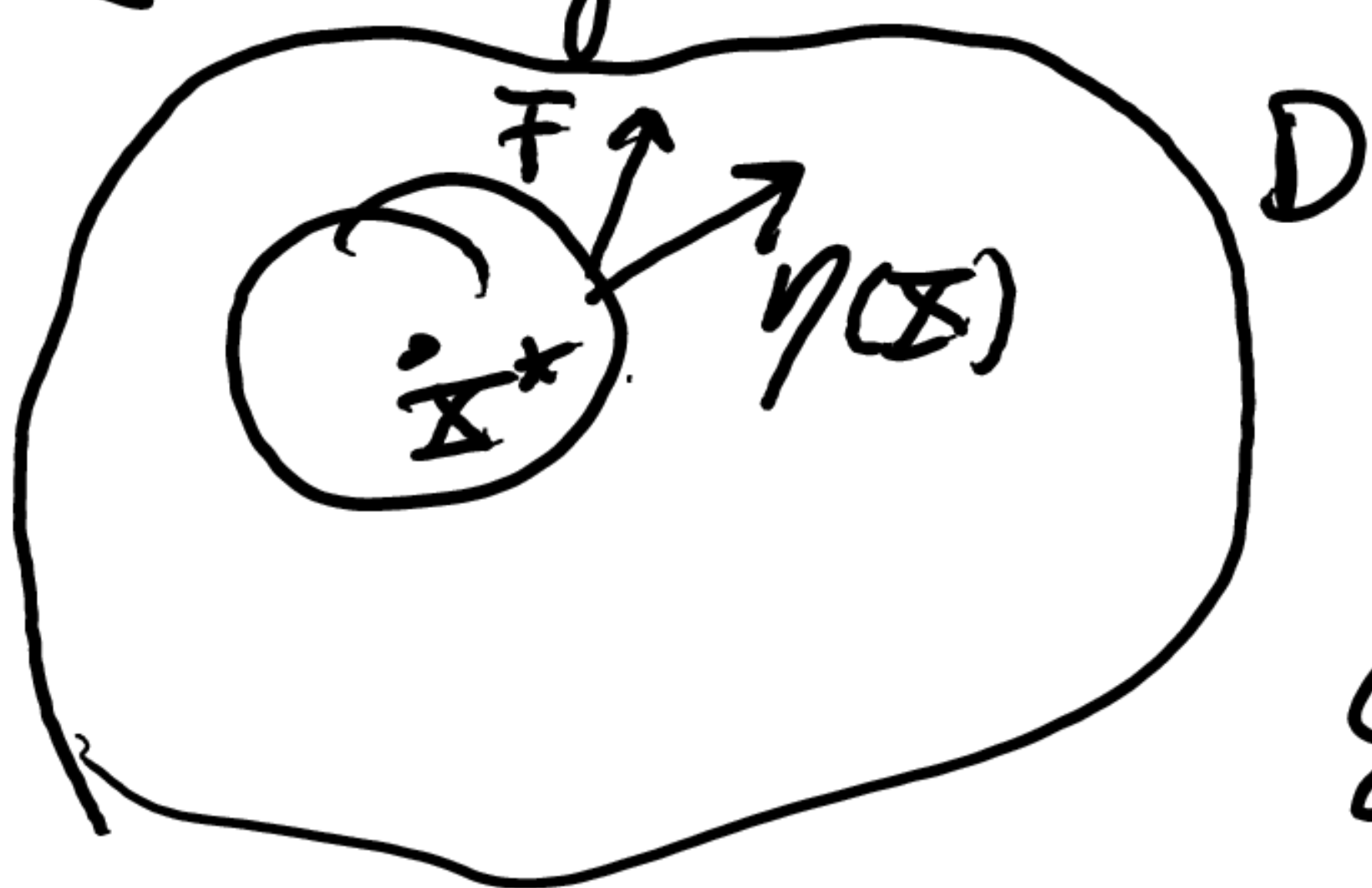
Corollary 1

Let $D \subset \mathbb{R}^2$ be compact and positively invariant for $\dot{X} = F(X)$.

If D contains a single EP that is a source, then D contains at least one limit cycle.

Proof idea Since X^* is a source,

$\exists$ neighborhood A s.t.



$$F(X) \cdot \eta(X) > 0 \quad \forall X \in \partial A$$

NB! strict inward

So $\tilde{D} = D \setminus A$ is

positively invariant. For each $X \in \partial A$, the closed orbit $\omega(X)$ does not contain X (since F points inward on all of ∂A). $\square$

Ex 3 Fritzsche-Nagumo

$$x' = x - \frac{x^3}{2} + y$$

$$y' = -x - by$$

with $b \in (\frac{1}{2}, 1)$.

Observe $D = [-10, 10] \times [-40, 40]$

is positively invariant as

$$f_1(-10, y) > 400, \quad f_1(10, y) < -400 \quad \forall y \in [-40, 40]$$

$$f_2(x, -40) > 10, \quad f_2(x, 40) < -10 \quad \forall x \in [-10, 10]$$

D contains one EP as

$$x = -by \text{ \& } x - \frac{x^3}{2} + y = 0 \Rightarrow y((1-b) + b^3 y^2) = 0$$

$$\Rightarrow (x, y) = (0, 0).$$

$$DF(0) = \begin{bmatrix} 1 & 1 \\ -1 & -b \end{bmatrix}$$

$$\lambda^2 + (1-b)\lambda + (1-b) = 0$$
$$\lambda_{\pm} = \frac{1-b \pm \sqrt{(1-b)^2 - 4(1-b)}}{2}$$

has $\operatorname{Re}(\lambda_{\pm}) > 0 \Rightarrow 0$ is a source.

By Corollary 1, D contains a limit cycle. □

Not curriculum: Alternative more technical verification that D contains a periodic solution using Thm 1 instead of Cor 1:

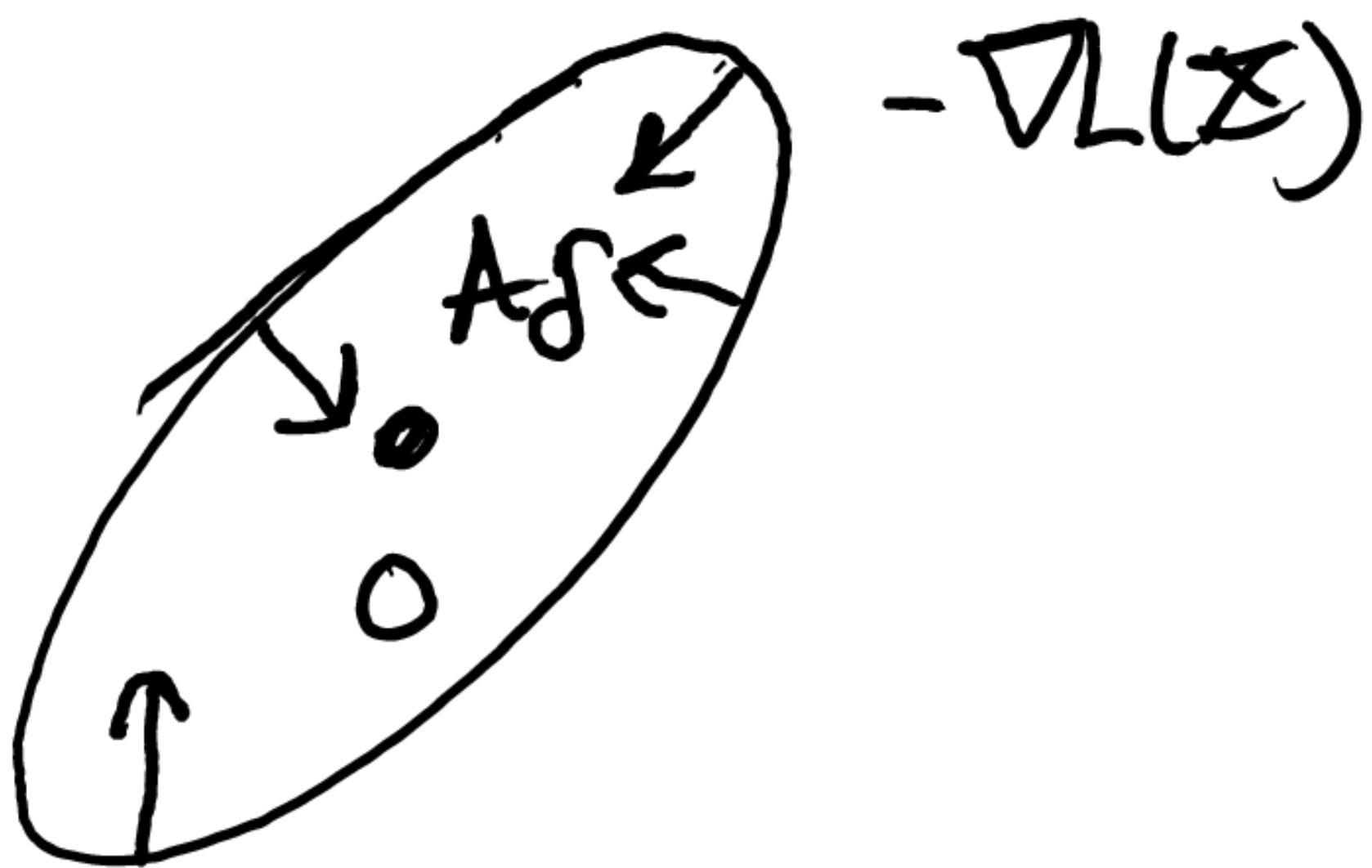
Consider $L(x, y) = x^2 + bxy + y^2 \geq \frac{1-b}{2}(x^2 + y^2)$.
It has a strict minimum at 0 .

For small $\delta > 0$, consider

$$A_\delta := \{(x, y) \mid L(x, y) < \delta\}.$$

Then $0 \in A_\delta$ and A_δ is a neighborhood of 0 (as 0 is a strict minimum).

For $\mathbb{X} \in \partial A_\delta = L^{-1}(\delta)$, $-\nabla L(\mathbb{X})$ is parallel to inward pointing normal $\eta(\mathbb{X})$.



$\Rightarrow F(\underline{x}) \cdot \eta(\underline{x})$ has same sign as

$$F(\underline{x}) \cdot (-\nabla L(\underline{x})) = \left(DF(0)\underline{x} + O(|\underline{x}|^2) \right) \cdot \underbrace{(-\nabla L(\underline{x}))}_{O(|\underline{x}|)}$$
$$= \begin{bmatrix} x+y \\ -x-by \end{bmatrix} \cdot \begin{bmatrix} -2x-(1+b)y \\ -(1+b)x-2y \end{bmatrix} + O(|\underline{x}|^3)$$

$$= -2x^2 - (1+b)y^2 - (3+b)xy + (1+b)x^2 + 2by^2 + (2+b+b^2)xy + O(|\underline{x}|^3)$$

$$= (b-1)(x^2 + y^2) + (b^2-1)xy + O(|\underline{x}|^3)$$

$$\leq \frac{-(b-1)^2}{2}(x^2 + y^2) + O(|\underline{x}|^3) \leq 0$$

for all $\underline{x}$ on $L^{-1}(f)$ for $\delta > 0$
sufficiently small. Hence

$\tilde{D} = D \setminus A_f$ is positively invariant.