

MAT3440 Stability of Equilibrium points

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Consider the ODE $X' = F(X)$ with a smooth vector field $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and an equilibrium point (EP) X^* . Then the EP is:

- **stable** if for any $\delta > 0$ there exists a $\delta_1 \in (0, \delta)$ s.t.

$$|X(t; X_0) - X^*| < \delta \quad \text{for all } X_0 \in B_{\delta_1}(X^*) \text{ and } t \geq 0.$$

- **asymptotically stable** if it is stable and there exists a $\delta > 0$ s.t.

$$\lim_{t \rightarrow \infty} X(t; X_0) = X^*$$

- **unstable** if it is not stable.

Sinks, sources and saddles for the linear setting

Let $F(X) = AX$ for some hyperbolic matrix $A \in \mathbb{R}^{n \times n}$ with eigenvalues $\lambda_1, \dots, \lambda_n$. Then

- X^* is a **sink** if $\operatorname{Re}(\lambda_i) < 0$ for all $i = 1, \dots, n$. For linear problems, X^* is a sink $\iff X^*$ is asymptotically stable.
- X^* is a **saddle** if $\operatorname{Re}(\lambda_i) < 0$ for at least one eigenvalue and $\operatorname{Re}(\lambda_i) > 0$ for at least one eigenvalue.
- X^* is a **source** if $\operatorname{Re}(\lambda_i) > 0$ for all $i = 1, \dots, n$.

Sinks, sources and saddles for the nonlinear setting

For $X' = F(X)$ with the EP X^* , assume that the Jacobian $DF(X^*)$ is hyperbolic with $\lambda_1, \dots, \lambda_n$. Then, thanks to the linearization theorem, the theory for sinks sources etc carries over:

- X^* is a **sink** if $\operatorname{Re}(\lambda_i) < 0$ for all $i = 1, \dots, n$.
- **Note:** For nonlinear ODEs, X^* is a sink $\implies X^*$ is asymptotically stable, but, unlike for linear ODEs, there is no equivalence between being a sink and asymptotically stable!
- X^* is a **saddle** if $\operatorname{Re}(\lambda_i) < 0$ for at least one eigenvalue and $\operatorname{Re}(\lambda_i) > 0$ for at least one eigenvalue.
- X^* is a **source** if $\operatorname{Re}(\lambda_i) > 0$ for all $i = 1, \dots, n$.

Example This is a restrictive definition, as it excludes classifying the EP 0 of the dynamics

$$x' = -x^3$$

to be a sink as the Jacobian $f'(0) = 0$. (It would be quite natural to call it a sink, since any solution will move directly, with non-zero velocity, toward 0.) The benefit of the definition is however that it is easy to apply to classify a point, it is the one used in the textbook, so we will stick to it in this course.