

Mat 3440 Stiff ODE

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Implicit RK method

An s -stage RK method with the tableau (1)

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|ccc} c_1 & a_{1,1} & \cdots & a_{1,s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s,1} & \cdots & a_{s,s-1} \\ \hline & b_1 & \cdots & b_s \end{array} \quad (1)$$

is called **implicit** if $a_{i,j} \neq 0$ for at least one component with $j \geq i$.

s-stage RK method

Recall that an s -stage RK method with Butcher tableau (1) has the stepping rule

$$\bar{X}_{n+1} = \bar{X}_n + h\Phi(\bar{X}_n, \bar{X}_{n+1}, t_n, h)$$

with

$$\Phi(\bar{X}_n, \bar{X}_{n+1}, t_n, h) = \sum_{i=1}^s b_i k_i,$$

and system of equations

$$k_i = F\left(t_n + c_i h, \bar{X}_n + h \sum_{j=1}^s a_{ij} k_j\right) \quad i = 1, \dots, s.$$

Stability properties of RK methods

- Implicit methods tend to be more stable than explicit methods.
- This means that one can often solve problems with larger $h > 0$ with implicit methods than explicit.
- Implicit methods are more suitable for **stiff problems**, involving.
- Drawback with implicit methods: typically more computationally costly than explicit methods, as one needs to solve implicit equation for $\tilde{X}_{n+1}$ every iteration.

Test problem

$$x' = -10x, \quad x(0) = 1.$$

Stability regions

$$|R_{FE}(-10h)| = |1 - 10h| \leq 1 \iff h \leq 2/10$$

$$|R_{IE}(-10h)| = |1 + 10h| \geq 1 \quad \text{holds for any } h > 0$$

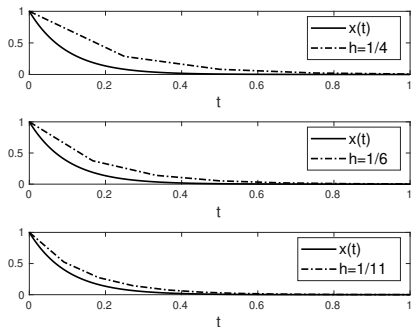
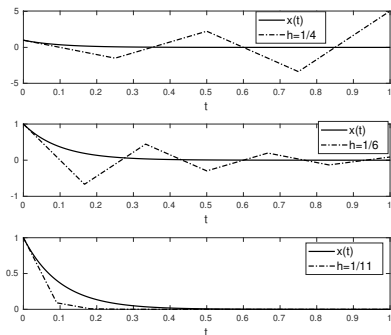


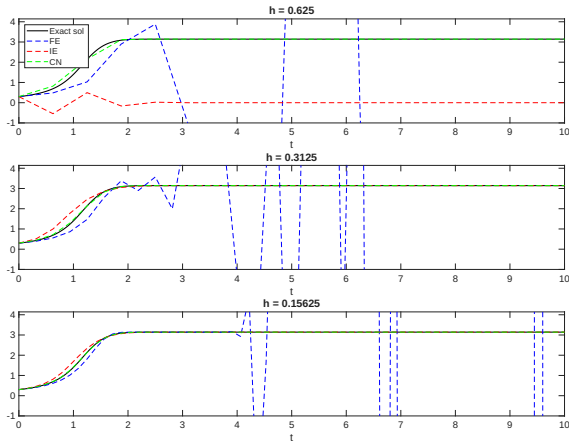
Figure: **Left:** Explicit Euler. **Right:** Implicit Euler.

Nonlinear stiff problem – experimental study

$$x' = e^t \sin(x) \quad t \in [0, 10]$$
$$x(0) = 0.3$$

Exact bounded solution: $x(t) = 2 \tan^{-1} \left[\tan(x_0/2) \exp(\exp(t) - 1) \right]$

Numerical test:



Linear stability and absolute stability when $d > 1$

- **Definition** For any ODE $X' = AX$ with a symmetric matrix $A = VDV^T$, the ODE is called **linearly stable** if $\operatorname{Re}(\lambda) \leq 0$ for all EV $\lambda_1, \dots, \lambda_d$ of A .
- This is equivalent to $X(t)$ being bounded in norm for all $t \geq 0$ and any $X_0 \in \mathbb{R}^d$.
- **Stability criterion for RK methods in higher dimensions:** Let $R(z)$ be the stability function of an RK method. Then $|\bar{X}_n|$ is bounded for all $n \geq 0$ if $h > 0$ is chosen such that $\|R(Ah)\|_2 \leq 1$.
- For any symmetric matrix A , stability of an RK method is equivalent to (by the spectral theorem for symmetric matrices):

$$\max_{i \in \{1, \dots, d\}} |R(\lambda_i h)| \leq 1.$$

Stiff problem in higher dimensions

Can for example be a linear problem with dynamics on different timescales (combined with Forward Euler):

$$\dot{X} = \begin{pmatrix} -1 & 0.1 \\ 0.1 & -100 \end{pmatrix} X,$$

where one can show that $x_2(t)$ varies on a faster timescale than $x_1(t)$.

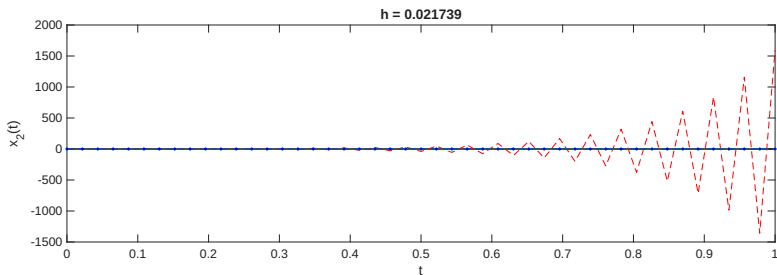
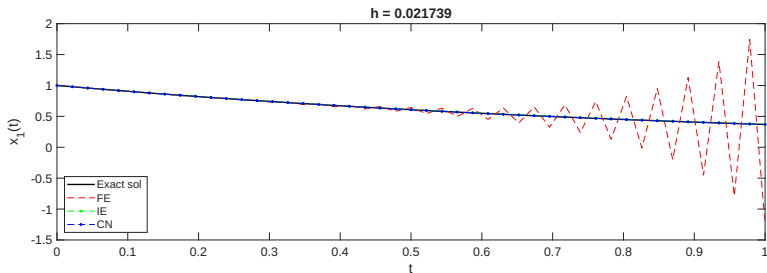
EV: $\lambda_1 \approx -1$, $\lambda_2 \approx -100$.

Timestep $h > 0$ ensuring stability for Forward Euler

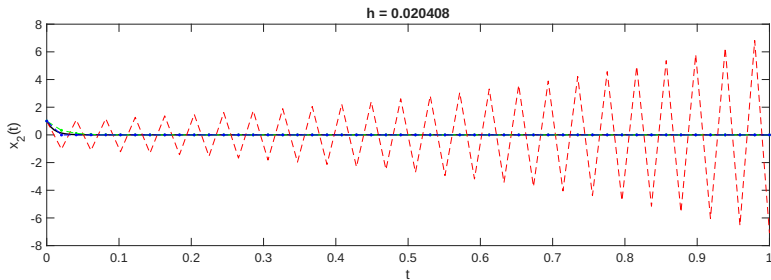
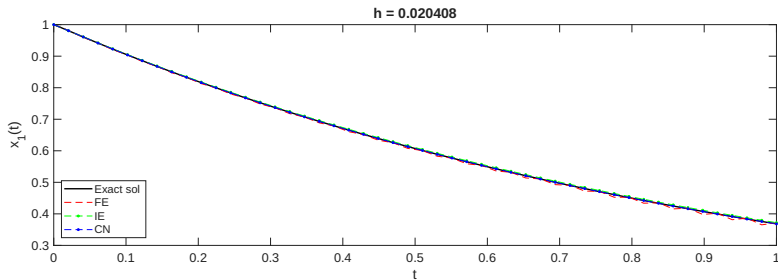
$$\max_{i \in \{1, \dots, d\}} |1 + \lambda_i h| \leq 1 \iff |1 - 100h| \leq 1 \iff h \leq 1/50.$$

Numerical test $h = 1/46$

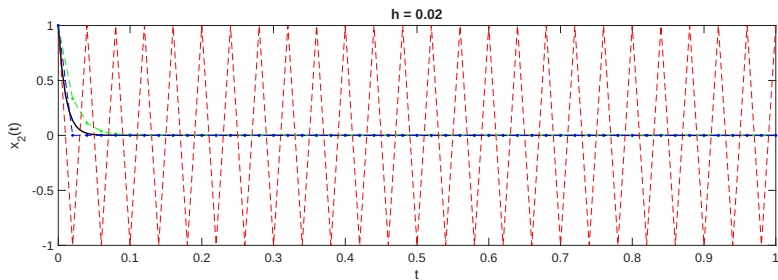
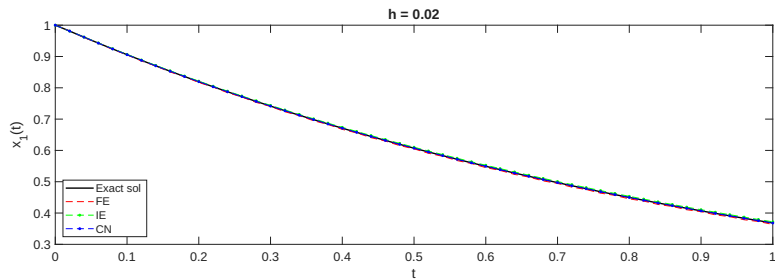
FE is unstable while IE and CN are stable:



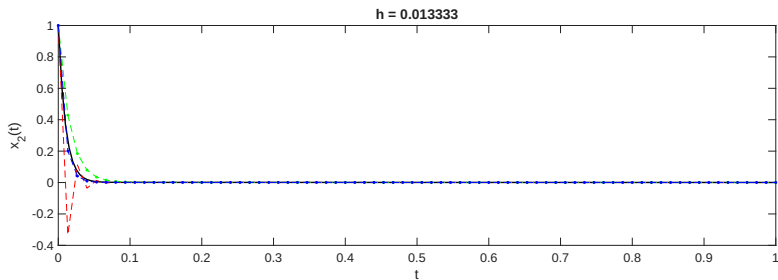
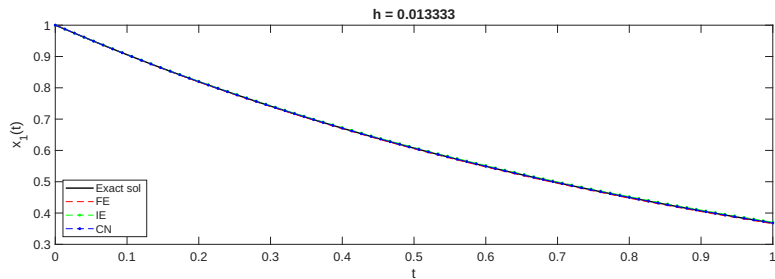
Numerical test $h = 1/49$



Numerical test $h = 1/50$



Numerical test $h = 1/75$



Stability functions for explicit RK methods

For a certain class of explicit RK methods a nice pattern emerges: Let us call Forward Euler RK1 and recall its tableau:

$$\begin{array}{c|c} 0 & \\ \hline & 1 \end{array} \quad \text{with} \quad R(z) = 1 + z$$

Let us call Heun's method RK2 and recall its tableau:

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \quad \text{where} \quad R(z) = 1 + z + z^2/2$$

We call the following method RK3

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 1 & -1 & 2 & \\ \hline & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \end{array} \quad \text{where} \quad R(z) = 1 + z + z^2/2 + z^3/6$$

Runge-Kutta 4 has stability function $R(z) = \sum_{n=0}^4 \frac{z^n}{n!}$.

To estimate the stability regions of these methods

$$S = \{z \in \mathbb{C} \mid |R(z)| \leq 1\}$$

we can get help from Matlab:

```
RK2 = @(z) 1+z +z.^2/2;  
RK3 = @(z) 1+z +z.^2/2 + z.^3/6;  
[X,Y] = meshgrid(-6:0.01:6,-6:0.01:6);  
contour(X,Y,abs(RK2(z)),[0 1], 'r', 'linewidth',1.4);  
contour(X,Y,abs(RK3(z)),[0 1], 'r', 'linewidth',1.4);
```

Explicit Runge-Kutta methods

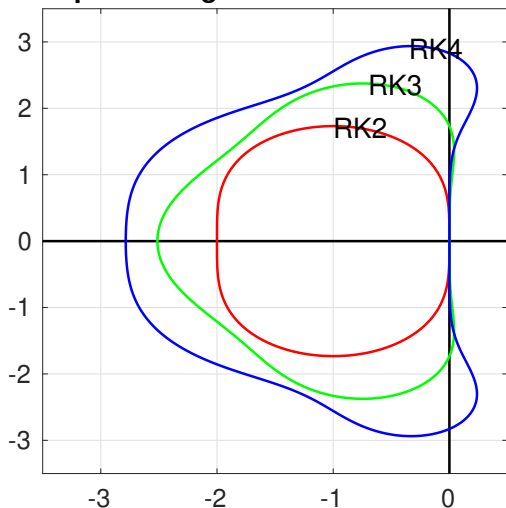


Figure: Region of absolute stability for RK 2,3 and 4 is the **interior** of the respective curves.