UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in MAT3500 — Topology

Day of examination: Friday December 17th 2010

Examination hours: 14.30 – 18.30

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Each subproblem (1a, 1b, etc.) carries the same weight.

Problem 1

1a

Let $\mathcal{B}$ be a collection of subsets of a given set X.

State the conditions that must be satisfied for $\mathcal{B}$ to be called a basis for a topology on X.

1b

Let $\mathcal{B}$ be the collection of subsets of $X = \mathbb{R}$ consisting of (1) the intervals $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ with a and b in $\mathbb{R}$, and (2) the singleton sets $\{c\}$ with $c < 0$ in $\mathbb{R}$.

Prove that $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}$.

1c

Let Y denote $\mathbb{R}$ with the topology $\mathcal{T}$. Let $f \colon Y \to Y$ be the bijection given by $f(x) = x + 2$ for $x \in Y$.

Prove that f is continuous.

1d

Let $g = f^{-1} \colon Y \to Y$ be the given by $g(x) = x - 2$ for $x \in Y$.

Prove that g is not continuous.

Problem 2

A topological space X is said to be locally compact if for each $x \in X$ there is a compact subspace C of X that contains a neighborhood U of x in X.

(Continued on page 2.)

2a

Let A be an open subspace of a locally compact Hausdorff space X.

Prove that A is locally compact.

(You may use the facts that a compact Hausdorff space is regular, and that a closed subspace of a compact space is compact.)

2b

Let ∞ be a point that is not in X, and give $Y = X \cup \{\infty\}$ the topology consisting of (1) all open subsets U of X, and (2) all sets of the form $Y - C$, where C is a compact subspace of X. Similarly, give $B = A \cup \{\infty\}$ the topology consisting of (1) all open subsets V of A, and (2) all sets of the form $B - K$, where K is a compact subspace of A. Let the function $f\colon Y \to B$ be defined by

$$f(x) = \begin{cases} x & \text{for } x \in A\,, \\ \infty & \text{for } x \in Y - A\,. \end{cases}$$

Prove that f is continuous.

Problem 3

3a

Give $\mathbb{C}$ and $\mathbb{C}^2$ the standard topologies, and consider $\mathbb{R} \subset \mathbb{C}$ as a subspace in the usual way. Let $U = \mathbb{C} - \{0\}$, $V = \mathbb{C} - \{0, 1\}$ and $T = \mathbb{C} - \mathbb{R}$ be subspaces of $\mathbb{C}$.

Describe the path components of each of the four spaces U, V, T and $V - T$.

3b

Let

$$X = \{(z, w) \in \mathbb{C}^2 \mid 0 \neq z \neq w \neq 0\}$$

be the subspace of $\mathbb{C}^2$ consisting of pairs (z, w) such that 0, z and w are three pairwise distinct points in $\mathbb{C}$. Let

$$f\colon X \to U \times V$$

be given by $f(z, w) = (z, w/z)$ for $(z, w) \in X$.

Prove that f is a homeomorphism, by exhibiting an inverse and showing that it is well-defined and continuous.

3c

Let

$$Y = \{(z, w) \in X \mid w/z \notin \mathbb{R}\}$$

be the subspace of $X \subset \mathbb{C}^2$ consisting of pairs (z, w) such that 0, z and w do not lie on the same (real) line in $\mathbb{C}$. Let $Z = X - Y$.

Determine the number of path components in each of the three spaces X, Y and Z.

(Continued on page 3.)

3d

Let $x_0 \in X$ and $f(x_0) = (u_0, v_0)$. Take as known that $\pi_1(U, u_0) \cong \mathbb{Z}$ and $\pi_1(V, v_0) \cong F_2$, where $\mathbb{Z}$ is the additive group of integers and F_2 is the free, non-abelian group on two generators.

Express the fundamental group $\pi_1(X, x_0)$ in terms of $\mathbb{Z}$ and F_2. Is $\pi_1(X, x_0)$ an abelian group?

SLUTT