Proposed solutions for MAT3500/4500 fall 2010

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Problem 1a

The following two conditions must be satisfied:
(1) For each $x \in X$ there is a $B \in \mathcal{B}$ with $x \in B$.
(2) If $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$ then there is a $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$.

Problem 1b

Condition (1) is clear: each $x \in \mathbb{R}$ is contained in the interval $(x-1, x+1) \in \mathcal{B}$.

We prove condition (2): Let $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$. We claim that $B_3 = B_1 \cap B_2$ is in $\mathcal{B}$; then clearly $x \in B_3 \subset B_1 \cap B_2$. If $B_1 = (a_1, b_1)$ and $B_2 = (a_2, b_2)$ then $B_3 = (a_3, b_3)$ where $a_3 = \max\{a_1, a_2\}$ and $b_3 = \min\{b_1, b_2\}$. Otherwise $B_1 = \{x\}$ or $B_2 = \{x\}$, so $B_3 = \{x\}$. In either case, $B_3 \in \mathcal{B}$.

Problem 1c

We must prove that $f^{-1}(B)$ is in $\mathcal{T}$ for each basis element $B \in \mathcal{B}$. If $B = (a, b)$ then $f^{-1}(B) = (a-2, b-2) \in \mathcal{B}$. If $B = \{c\}$ with $c < 0$ then $f^{-1}(B) = \{c-2\} \in \mathcal{B}$, since $c - 2 < 0$.

Problem 1d

Consider the basis element $B = \{-1\}$. Then $g^{-1}(B) = \{1\}$ is not in $\mathcal{T}$, since this singleton set is not a union of basis elements from $\mathcal{B}$. Hence g is not continuous.

Problem 2a

The spaces U and V are path connected. The space T has two path components, given by the complex numbers $z = x + iy$ with imaginary part $y > 0$ and $y < 0$, respectively. The space $V - T = \mathbb{R} - \{0, 1\}$ has three path components, given by the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$.

Problem 2b

To find the inverse function $g = f^{-1} \colon U \times V \to X$ we solve the equation $f(z, w) = (z, w/z) = (u, v)$ to get $(z, w) = (u, uv) = g(u, v)$. The function $g \colon U \times V \to X$ given by $g(u, v) = (u, uv)$ for $u \in U$ and $v \in V$ is well-defined, since $u \neq 0$ and $v \neq 0, 1$ implies that $0 \neq u \neq uv \neq 0$. It is continuous, since it is obtained from the map $\mathbb{C} \times \mathbb{C} \to \mathbb{C}^2$ taking (u, v) to (u, uv) by restriction to the subspace $U \times V \subset \mathbb{C} \times \mathbb{C}$ and "corestriction" to the image $X \subset \mathbb{C}^2$.

Problem 2c

The homeomorphism $X \cong U \times V$ restricts to homeomorphisms $Y \cong U \times T$ and $Z = X - Y \cong U \times (V - T)$. Since U is path connected, and V, T and $V - T$ have one, two and three path components, it follows that X, Y and Z have one, two and three path components, respectively.

Problem 2d

The homeomorphism $f\colon X \to U \times V$ induces an isomorphism $f_*\colon \pi_1(X, x_0) \cong \pi_1(U \times V, (u_0, v_0))$ and the projections $U \times V \to U$ and $U \times V \to V$ induce an isomorphism

$$\Phi\colon \pi_1(U \times V, (u_0, v_0)) \cong \pi_1(U, u_0) \times \pi_1(V, v_0)\,.$$

We are given that $\pi_1(U, u_0) \cong \mathbb{Z}$ and $\pi_1(V, v_0) \cong F_2$, so $\pi_1(X, x_0) \cong \mathbb{Z} \times F_2$. This is not an abelian group, since it has F_2 as a non-abelian subgroup.

Problem 3a

Let $x \in A \subset X$ be any point. By local compactness of X there is a compact subspace C of X that contains a neighborhood U of x in X. Since A is open in X, so is the intersection $A \cap U \subset C$. Hence $A \cap U$ is a neighborhood of x in the compact Hausdorff space C. By regularity of C there is a neighborhood V of x in C, with closure $\bar{V}$ in C contained in $A \cap U$. Here $\bar{V}$ is compact, since it is a closed subset of the compact set C. Furthermore, V is open in $A \cap U$, and $A \cap U$ is open in A, so V is a neighborhood of x in A. Hence A is locally compact.

Problem 3b

Let $V \subset B$ be open. If V is an open subset of A, then $\infty \notin V$ and $f^{-1}(V) = V$. Since V is open in A and A is open in X, it follows that V is an open subset of X. Hence $f^{-1}(V)$ is open in Y. Otherwise, $V = B - K$ where $K \subset A$ is compact. Then $f^{-1}(V) = Y - K$, where $K \subset X$ is compact. Hence $f^{-1}(V)$ is open in Y, also in this case. Thus f is continuous.