UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT3500/4500 — Topology

Day of examination: Wednesday December 18th 2013

Examination hours: 09.00 – 13.00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Each of the 10 subproblems (1a, 1b, . . . , 5b) carries the same weight. You may use the results of an earlier subproblem to answer later questions, even if you have not answered the earlier one.

Problem 1

1a

Give the definitions of *regular spaces* and *normal spaces*, and state the *Urysohn metrization theorem.*

1b

Let $\mathbb{R}$ have the topology generated from the basis consisting of the half open intervals $[p, q)$ where p and q are rational numbers. Show that the sets $[p, q)$ form a basis, and that the basis elements are both closed and open in this topology. Describe the connected components of $\mathbb{R}$ with this topology.

1c

Explain how we can use the Urysohn metrization theorem to prove that $\mathbb{R}$ with the topology from Problem 1b is metrizable.

Problem 2

Let $\mathcal{T}_0$ be the standard topology on $\mathbb{R}^2$.

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2a

If $a < b \in \mathbb{R}$ and $0 < c \in \mathbb{R}$, let

$$B_{a,b,c} = \{(x, y) \in \mathbb{R}^2 \mid a < x < b \text{ and } |xy| < c\}.$$

Let $\mathcal{B}$ be the family of sets $B_{a,b,c}$. This will be a basis of a topology $\mathcal{T}_1$ on $\mathbb{R}^2$ (you do not have to prove this).
Explain why $\mathcal{T}_0$ is finer than $\mathcal{T}_1$.

2b

Is $\mathcal{T}_1$ from Problem 2a Hausdorff?
Show that every subset of the y-axis

$$Y = \{(0, y) \mid y \in \mathbb{R}\}$$

is compact in $\mathcal{T}_1$, but that there are only two closed subsets of Y in this topology.

Problem 3

Let X be a nonempty locally compact Hausdorff space and let $Y = X \cup \{\infty\}$ be the one point compactification of X.
Assume that there is a continuous function $f : [0, 1) \to X$ such that

$$\{x \in [0, 1) \mid f(x) \in C\}$$

is compact for each compact C in X.
Show that if X is path connected, then Y is path connected.

Problem 4

A topological space X is called *sequential* if for every $U \subseteq X$ we have that U is open when the following is satisfied:

> For every sequence $\{x_n\}_{n \in \mathbb{N}}$ from X with a limit $x \in U$ we have some $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$.

4a

Let $p : X \to Y$ be a quotient map. Show that if X is sequential, then Y is sequential.

4b

Show that if X is *first countable*, that is, there is a countable basis at each point $x \in X$, then X is sequential.

(Continued on page 3.)

Problem 5

5a

Let B be figure 8 given as the union of the circles in $\mathbb{R}^2$ with radius 1 and centers in $(0,0)$ and $(2,0)$.
Let $p : \mathbb{R} \to B$ be defined by

- $p(x) = (\cos 2\pi x, \sin 2\pi x)$ if $x \in [2a, 2a+1]$ for some $a \in \mathbb{Z}$.
- $p(x) = (2 - \cos 2\pi x, -\sin 2\pi x)$ if $x \in [2a+1, 2a+2]$ for some $a \in \mathbb{Z}$.

Show that p is not a covering map and give an example of a loop in B with no lifting to a path in $\mathbb{R}$.

5b

We now let $\equiv$ be the smallest equivalence relation on B where $(x,y) \equiv (v,w)$ when $y = w = 0$, $x = -1$ and $v = 3$.
Let E be the quotient space of B induced by this relation.
Find a covering map $q : E \to B$.

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