MAT3500/4500-2013-SOLUTIONS

Problem 1

a) Regular spaces and normal spaces will be T_1-spaces with the following extra properties:

- A space is regular if we for every closed set A and point $x \notin A$ have disjoint open sets U and V with $A \subseteq U$ and $x \in V$.
- A space is normal if we for every pair A, B of disjoint closed sets have disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.

The Urysohn metrization theorem states that if X is a second countable regular space, then it is metrizable, i.e. there is a metric on X inducing the topology on X.

b) Since the non empty intersection of two half open intervals $[p, q)$ and $[r, s)$ is a half open interval, the set of such intervals form a basis.
Each basis element $[p, q)$ will be closed since the complement is the union of all intervals $[q - n, q - n + 1)$ and $[p + n - 1, p + n)$ where $n \in \mathbb{N}$.
Since all distinct points can be separated by sets that are both closed and open, the connected components will consist of one point each.

c) The topology is T_1 since the connected components are singletons and connected components are always closed.
Then $\mathbb{R}$ with this topology must be regular, since if A is closed and $x \notin A$, there is a neighborurhood of X disjoint from A that is both closed and open, and the complement of this will be the desired open set around A.
Since $\mathbb{Q}$ is countable, the basis is countable. Then all assumptions in the Urysohn metrization theorem is satisfied, and X is metrizable.

Problem 2

a) If $B_{a,b,c}$ and B_{a_1,b_1,c_1} are two elements in $\mathcal{B}$ with a nonempty intersection, the two intervals (a, b) and (a_1, b_1) overlap. Let $(a_2, b_2) = (a, b) \cap (a_1, b_1)$ and let $c_2 = \min\{c, c_1\}$. Then

$$B_{a,b,c} \cap B_{a_1,b_1,c_1} = B_{a_2,b_2,c_2}.$$

Thus the non-empty intersection of two elements of $\mathcal{B}$ is an element of $\mathcal{B}$, and this ensures that $\mathcal{B}$ is a basis.
Each $B_{a,b,c}$ is an open region bounded by the lines $x = a$, $x = b$ and the graphs of the continuous functions $y = \frac{c}{|x|}$ and $y = -\frac{c}{|x|}$.

b) Every open set U in the topology $\mathcal{T}_1$ with $(0, y) \in U$ for some $y \in \mathbb{R}$ will have Y as a subset. Thus distinct points on the y-axis cannot be separated by disjoint open sets, and the topology is not Hausdorff.
The fact that every open set intersecting Y will have Y as a subset shows that all subsets of Y are compact, since every open covering will have a subcovering consisting of one set only.
Y is closed since the complement is the union of all sets $B_{a,b,c}$ where either $a > 0$ o r$b < 0$.
The only closed proper subset of Y will be the empty set, again since every open set intersecting Y will have Y as a subset.

Problem 3

a) If U is open and $\{x_n\}_{n\in\mathbb{N}}$ is a sequence with limit $x \in U$, then $x_n \in U$ for all but finitely many n by the definition of a limit of a sequence.
Now assume that U has this property and let $x \in U$. We will see that there must be a basis element B with $x \in B \subseteq U$. Assume that this is not the case. Then there is no open set V with $x \in V \subseteq U$. Let $\{B_n \mid n \in \mathbb{N}\}$ be the set of all basis elements with $x \in B_n$. Let

$$x_n \in \bigcap_{i=1}^{n} B_i - U$$

for each n. There will be one such x_n by our assumption, and by construction x is a limit of $\{x_n\}_{n\in\mathbb{N}}$. Then, on the one hand, $x_n \in U$ for all but finitely many $n$by the assumed property of U while $x_n \notin U$ for each n by construction of the sequence.
Thus the assumption that U contains no open neighborhood V of x leads to a contradiction, and must be false.

b) Let $U \subseteq Y$ be such that whenever $y \in U$ is a limit of the sequence $\{y_n\}_{n\in\mathbb{N}}$ then $y_n \in U$ for all except finitely many n.
We will prove that $V = p^{-1}(U)$ is open in X, and we will use that X is a sequential space.

So let $x \in V$ and let x be the limit of the sequence $\{x_n\}_{n\in\mathbb{N}}$. Let $y = p(x)$ and let $y_n = p(x_n)$.
Since p is continuous, y will be a limit of $\{y_n\}_{n\in\mathbb{N}}$, so $y_n \in U$ for all but finitely many n.
Then, by definition of V, $x_n \in V$ for all but finitely many n.
Then V is open, and U will be open in the quotient topology.

Problem 4

Since X is path connected, it is sufficient to find a path from one point in X to $*$.
Let f have the given properties. Let $g(1) = *$ and $g(x) = f(h(x)$ where $h(x) = \frac{x}{1-x}$ for $x \in [0,1)$. h is a monotonously increasing bijection between $[0,1)$ and $[0,\infty)]$ and g is clearly continuous on $[0,1)$.
It is sufficient to prove that g is continuous at $*$, so let V be an open neighbourhood of $*$.
Then $X - V$ is a compact, and by assumption, $\{x \in [0,\infty) \mid f(x) \in C$ is a compact.
Then $\{x \in [0,1) \mid g(x) \in C\}$ is a compact subset of $[0,1)$, so for some $a < 1$ we have that

$$1 \geq x > a \Rightarrow g(x) \in V.$$

This proves continuity in $*$.

Problem 5

Since there probably will be several solutions to this problem, we only give the sketches of the constructions and arguments.

a) p will not provide any even covering of a neighbourhood of $(1,0)$, the local "cross" is not homeomorphic to any open segment of $\mathbb{R}$.
The double loop $f(x) = (\cos 4\pi x, \sin 4\pi x)$ based at $(1,0)$ will not have a continuous lifting to a path $\tilde{f}$ with initial value 0 because a discontinuity is required at $x = \frac{1}{2}$.

b) E will have two "crosses" that we map, by q to the one cross in B.
We let q map the two upper semicircles in E to the left copy of S^1 in B and the two lower semicircles in E to the right copy of S^1 in B.Then, for each point in $b \in B$ there will an open neighbourhood U of b that is evenly covered by q, everywhere with two copies.