MAT 3500/4500

SUGGESTED SOLUTION

Problem 1

Let Y be a topological space. Let X be a non-empty set and let $f : X \to Y$ be a surjective map.

a) Consider the collection τ of all subsets U of X such that $U = f^{-1}(V)$ where V is open in Y. Show that τ is a topology on X. Also show that a subset $A \subset X$ is connected in this topology if and only if $f(A)$ is a connected subset of Y.

Solution: We have $X = f^{-1}(Y)$ and $\varnothing = f^{-1}(\varnothing)$ so X and $\varnothing$ are in τ. Consider $\{V_i \mid i \in I\}$ with each V_i open in Y. Then $\bigcup_{i\in I} f^{-1}(V_i) = f^{-1}(\bigcup_{i\in I}(V_i)$, and since $\bigcup_{i\in I}(V_i)$ is open in Y, τ is closed under arbitrary unions. Let $V_j, j = 1, \dots n$, be open sets in Y. Then $\bigcap_{j=1}^{n} f^{-1}(V_j) = f^{-1}(\bigcap_{j=1}^{n}(V_j)$, and since $\bigcap_{j=1}^{n}(V_j)$ is open in Y, τ is closed under finite intersections. Together this shows that τ is a topology on X.

It is clear from the definition of τ that f becomes continuous when X is given the topology τ. So if A is connected in τ, $f(A)$ becomes connected in Y.

Now let $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$ be sets in τ with V_1 and V_2 open sets in Y. Let $A \subset X$ with $A \cap U_1 \cap U_2 = \varnothing$, $A \subset U_1 \cup U_2$. Then we must have $f(A) \cap V_1 \cap V_2 = \varnothing$. For if $f(a) \in V_1 \cap V_2$ with $a \in A$, then $f^{-1}(f(a)) \subset f^{-1}(V_1) \cap f^{-1}(V_2)$ hence $a \in U_1 \cap U_2$. Assume $f(A)$ is connected. Since $A \subset U_1 \cup U_2$, we have that $f(A) \subset V_1 \cup V_2$, and since $f(A)$ is connected, we must either have that $f(A) \cap V_1 = \varnothing$ or $f(A) \cap V_2 = \varnothing$. Assuming $f(A) \cap V_1 = \varnothing$, we get that $A \cap U_1 = \varnothing$, and it follows that we cannot find a separation of A. It follows that A is connected.

b) Show that a subset K of X is compact in τ if and only if $f(K)$ is a compact subset of Y.

Solution: If K is compact in τ, $f(K)$ is compact in Y since f is continuous in τ. Assume $K \subset X$ with $f(K)$ compact in Y. Let $\{U_i\}$ be a collection sets in τ covering K. Then for each i, $U_i = f^{-1}(V_i)$ with V_i open in Y. So $\{V_i\}$ is an open covering of $f(K)$, and we may pick a finite subcollection covering $f(K)$. Let $V_1, \dots, V_n$ be such subcollection. Then $U_j = f^{-1}(V_j)$ is a finite subcollection of $\{U_i\}$, and we have $K \subset f^{-1}(f(K)) \subset \bigcup_{j=1}^{n} f^{-1}(V_j) = \bigcup_{j=1}^{n} U_j$. This shows that K is compact in τ.

c) Show that if $L \subset X$ is closed in τ then $f(L)$ is closed in Y and $L = f^{-1}(f(L)))$.
Let Z be a topological T_1-space (a space where one-point sets are closed). Let $h : X \to Z$ be a map. Show that h is continuous if and only if there exists a continuos map $k : Y \to Z$ such that $k \circ f = h$.

Solution: Note that if $U = f^{-1}(V)$ with V open in Y, then U is saturated with respect to f and we have that $f(U) = V$, hence $f(U)$ is open in Y when $U \in \tau$. Since the complement of a saturated set must be saturated, L is saturated if L is closed in τ. So $L = f^{-1}(f(L))$. Now $L = X - U$ with $U \in \tau$, and since U is saturated $f(L) = f(X - U) = Y - f(U)$. Since $f(U)$ is open in Y, it follows that $f(L)$ is closed in Y.

If there exists a continuous map $k : Y \to Z$ such that $k \circ f = h$, then h is a composition of continuous maps, hence continuos. Assume that h is continuous. There exits a map $k : Y \to Z$ such that $k \circ f = h$ if h is constant on the fibers of f. That is, if $y \in Y$ and $h \,|\, f^{-1}(y)$ is constant (for each y). We may then define k by $k(y) = h(x)$ when x is such that $f(x) = y$. To see that h is constant on the fibers of f, let $y \in Y$, $x \in X$ and $z \in Z$ be such that $f(x) = y$ and $h(x) = z$. Put $L = h^{-1}(z)$. Now since Z is T_1, $\{z\}$ is closed and since h is continuos, L is closed. From above we get that $L = f^{-1}(f(L))$. Since $x \in L$ and $y \in f(L)$, we must have $f^{-1}(y) \subset L = h^{-1}(z)$. It follows that h is constant on the fibers of f and we may define k as described above. To prove that k is continuous, let P be closed in Z. Then $h^{-1}(P) = f^{-1}(k^{-1}(P))$ is closed (since h is continuous), and from above it follows that $f(h^{-1}(L))$ is closed: Since f is surjective, we get that $f(h^{-1}(L)) = k^{-1}(P)$, so $k^{-1}(P)$ is closed, hence k is continuos.

d) Now let $Y = \mathbb{R}$ with standard topology. Let $X = \mathbb{R}^2$, and let f be the map given by $f(x, y) = xy$, where $(x, y) \in \mathbb{R}^2$. Let τ be the topology in $\mathbb{R}^2$ described in a) above. Explain why open sets, respectively closed sets in τ also are open sets, respectively closed sets in the standard topology of $\mathbb{R}^2$. Find an example of a compact set in τ which neither is closed nor bounded in the euclidean metric of $\mathbb{R}^2$. Also find an example of a disconnected set in the standard topology of $\mathbb{R}^2$ which is connected in τ.

It is clear that $f(x, y) = xy$ is also continuos when $\mathbb{R}^n$ is given the standard topology. So if V is open in $\mathbb{R}$, $f^{-1}(V)$ is open in the standard topology, hence all sets in τ are open in the standard topology. Since the closed sets in τ are complements of sets in τ hence complements of open sets in the standard topology, they are also closed in the standard topology.

If we let $K = \{(x, 0) \,|\, x > 0\}$, K is a set which obviously neither is closed nor bounded in the Euclidean metric. But $f(K) = \{0\}$, which obviously is compact in $\mathbb{R}$, so by (b) above, K must be compact in τ.

If $A = \{(-1, 0)\} \cup \{(1, 0)\}$, A is a set which obviously is not connected in the standard topology. But $f(A) = \{0\}$, which obviously is connected in $\mathbb{R}$, so by (a) above, A must be connected in τ.

Problem 2

Let X be a topological space. Suppose that, for every pair of points $x, y \in X$, $x \neq y$, there exists a continuous map $f : X \to [0,1]$ such that $f(x) = 1$ and $f(y) = 0$.

a) Show that X is a Hausdorff space.

Solution: Let $x, y \in X$, $x \neq y$, and $f : X \to [0,1]$ a continuos map such that $f(x) = 1$ and $f(y) = 0$. Let $U = f^{-1}((\frac{1}{2}, 1])$, and $V = f^{-1}([0, \frac{1}{2}))$. Then U and V are disjoint neighbourhoods of x and y respectively, hence X is a Hausdorff space.

b) Let K be a non-empty compact subset of X, $K \neq X$. Let $x \in X - K$. Show that there exits a continuous map $f : X \to [0,1]$ such that $f(x) = 1$ and $f(K) \subset [0, \frac{1}{2})$.

Solution: For each $y \in K$ let $f_y : X \to [0,1]$ be a continuous map such that $f_y(y) = 0$ and $f_y(x) = 1$. Let $U_y = f_y^{-1}([0, \frac{1}{2}))$. Then $\{U_y \,|\, y \in K\}$ is an open covering of K. Since K is compact, we may find a finite set $\{y_1, \ldots y_n\}$ of points in K such that $\{U_{y_1}, \ldots U_{y_n}\}$ covers K. Let $f_i = f_{y_i}$. Let f be the product $f = f_1 \cdot f_2 \cdots f_n$. If $y \in K$ then $y \in U_{y_i}$ for some i, so $0 \leq f_i(y) < \frac{1}{2}$, and since $0 \leq f_j(y) \leq 1$ when $j \neq i$, we must have that $f(y) \in [0, \frac{1}{2})$. Hence $f(K) \subset [0, \frac{1}{2})$.

Problem 3

A subset A of a topological space X is dense if $\bar{A} = X$.

a) Let U and V be open dense subsets of a space X. Show that $U \cap V$ is dense.

Solution: Let $x \in X$ and let W be a neighbourhood of x. Since U is dense, we can find $u \in W \cap U$. Now $W \cap U$ is a neighbourhood of u, so since V is dense, we can find $v \in W \cap U \cap V$. So every neighbourhood of x intersects $U \cap V$ and x is consequently in the closure of $U \cap V$. Since x was arbitrary, we get that $\overline{U \cap V} = X$, hence $U \cap V$ is dense.

b) Let (X, d) be a metric space, and suppose that X has a countable dense subset. Prove that X is second countable (that X has a countable basis for the metric topology).

Solution: Let A be a countable dense subset of X. Let $\mathcal{A}$ be the collection of balls $B_d(a, \frac{1}{n})$ where $a \in A$ and $n \in \mathbb{Z}_+$. Since $A \times \mathbb{Z}_+$ is countable, this collection is countable. We will prove that this collection is a basis for the topology. To this end, let U be open in the metric topology, and let $x \in U$. Then we may find $\epsilon > 0$ such that $B_d(x, \epsilon) \subset U$. Let $N \in \mathbb{Z}_+$ be such that $\frac{1}{N} < \frac{\epsilon}{2}$. Let $n > N$. Since A is dense, we can find $a \in A$ be such that $a \in B_d(x, \frac{1}{N})$. Let $y \in B_d(a, \frac{1}{N})$, then $d(x, y) \leq d(x, a) + d(a, y) < \frac{2}{N} < \epsilon$. This proves that $B_d(a, \frac{1}{N}) \subset B_d(x, \epsilon) \subset U$. Moreover since $d(x, a) < \frac{1}{N}$, we get that $x \in B_d(a, \frac{1}{N}) \subset U$. Since $B_d(a, \frac{1}{N})$ is a set in $\mathcal{A}$, this shows that $\mathcal{A}$ is a basis for the topology.

Problem 4

In this problem, you can use without proof that if $x \in \mathbb{S}^n$ (here $\mathbb{S}^n$ is the unit n-sphere in $\mathbb{R}^{n+1}$), then there exists a homeomorphism $h : \mathbb{S}^n - \{x\} \to \mathbb{R}^n$

a) Give the definition of a deformation retract A of a topological space X. Give an example of a space X and a proper subset A which is a deformation retract of X. Consider the space $Y = \mathbb{S}^n - \{p, q\}$ where $p = (0, \dots, 0, 1)$ and $q = (0, \dots, 0, -1)$. Let $x_0 = (1, 0, \dots, 0) \in Y$. Find $\pi_1(Y, y_0)$ for all $n > 1$.

Solution: For the definition of a deformation retract, see the text book. If $X = \mathbb{R}^n - \{0\}$, and $A = \mathbb{S}^{n-1}$, we can define a retraction $r(x) = \frac{x}{||x||}$. Then $H(x,t) = tr(x) + (1-t)x$ is a homotopy between the identity on X and $r(x)$, such that $H(x,t) = x$ when $x \in A$, showing that A is a deformation retract of X.

Now if $h : \mathbb{S}^n - \{p\} \to \mathbb{R}^n$ is a homeomorphism, then $Y = \mathbb{S}^n - \{p, q\}$ is mapped to $\mathbb{R}^n$ minus one point. So especially, X and Y above becomes homeomorphic. So the fundamental groups of X and Y are isomorphic. Since $A = \mathbb{S}^{n-1}$ is a deformation retract of X, the fundamental group of X and $\mathbb{S}^{n-1}$ are isomorphic. It follows that when $n = 2$ the fundamental group of Y is isomorphic to $\pi_1(\mathbb{S}^1)$, hence isomorphic to the additive group $\mathbb{Z}$. When $n > 2$, $\pi_1(\mathbb{S}^{n-1})$ is trivial, hence $\pi_1(Y, y_0)$ is the trivial group (the group with one element).

b) Let X be a topological space and $f : X \to \mathbb{S}^n$ be a continuous map, $n \geq 1$. Prove that f is nullhomotopic if f is not surjective.

Solution: Since f is not surjective, we can find $x \in \mathbb{S}^n$ such that $x \notin f(X)$. Let $h : \mathbb{S}^n - \{x\} \to \mathbb{R}^n$ be a homemorphism. Then we may define $h \circ f$, which is a continous map from X into a contractible space $\mathbb{R}^n$. Since any continuous map into a contractible space is nullhomotopic, it follows that $h \circ f$ is nullhomotopic and we get that $f^{-1} \circ (h \circ f) = f$ is also nullhomotopic (here we use the fact that if two maps are homotopic, the two induced maps we get by composing each with a third map are also homotopic, and that a composition with a constant map is a constant map).

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