Calculus and Counterexamples

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Limits in high school mathematics

- To differentiate polynomials, you only need algebra to compute limits.
- \( \lim_{x \to 0} \frac{\sin(x)}{x} = 1. \)
- Definition of \( e. \)
Definition of $e$

▶ Does $s_n = \left(1 + \frac{1}{n}\right)^n$ converge?
▶ We want to use the fact that a bounded and increasing sequence converges, but it is not clear that $s_n$ is either bounded or increasing.
▶ The binomial formula shows that

\[
s_n = \left(1 + \frac{1}{n}\right)^n = 1 + \frac{1}{n} + \frac{n(n-1)}{2!} \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \frac{1}{n^3} + \cdots + \frac{n(n-1)(n-2) \cdots 1}{n!} \frac{1}{n^n}
\]

\[
= 1 + \frac{1}{1!} + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right).
\]

Definition of $e$ 2

▶ The product is hard to analyze, since the number of factors increase, while the factors themselves decrease. However, the binomial formula converts $s_n$ to a sum of $n$ terms.
▶ Since all the terms in the parenthesis are positive, we have now written $s_n$ as a sum of $n$ positive terms. When we go from $s_n$ to $s_{n+1}$, terms of the form $\left(1 - 1/n\right)$ will change to $\left(1 - 1/(n+1)\right)$, which is larger. So the first $n$ terms increase, and we also add another positive term. It is therefore clear that $s_n$ is increasing.
Definition of e

Consider the series \( \sum_{k=0}^{\infty} \frac{1}{k!} \) with partial sums
\[
 t_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.
\]

Since \( t_n \) is obtained from \( s_n \) by removing the parenthesis, and all the terms in the parenthesis are less than 1, we see that \( s_n \leq t_n \). Since going from \( t_n \) to \( t_{n+1} \) just adds a positive term, we see that \( t_n \) is also increasing.

Since
\[
 n! = 1 \cdot 2 \cdot 3 \cdots n > 1 \cdot 2 \cdot 2 \cdots 2 = 2^{n-1},
\]
we have
\[
 s_n < 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3.
\]

It follows that \( s_n \) is bounded and increasing, so \( e \) exists and \( e \leq 3 \).

Continuity

\( f : U \to \mathbb{R} \) is continuous at \( a \in U \) if \( \lim_{x \to a} f(x) = f(a) \) and continuous on \( U \) if it is continuous at all points in \( U \).

Some people say that \( f \) is continuous if and only if we can draw the graph of \( f \) without lifting the pen. However, \( f(x) = \frac{1}{x} \) is continuous on \( U = \mathbb{R} - \{0\} \).
Product rule

\[ f(x + \Delta x)g(x + \Delta x) - f(x)g(x) = (f(x + \Delta x) - f(x))g(x) \]
\[ + (g(x + \Delta x) - g(x))f(x) \]
\[ + (f(x + \Delta x) - f(x))(g(x + \Delta x) - g(x)) \]

Source of counterexamples

\[ f_n(x) = \begin{cases} 
  x^n \sin(1/x) & \text{if } x \neq 0, \\
  0 & \text{if } x = 0. 
\end{cases} \]

\[ f_0 \text{ is not continuous, since } \lim_{x \to 0} f_0(x) \text{ does not exist. However,} \]
\[ \lim_{x \to 0} f_1(x) = 0, \text{ so } f_1 \text{ is continuous.} \]
\[ f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \]

\[ f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \]

**Monotonicity**

- Mean Value Theorem: Assume that \( f \) is differentiable on \((a, b)\) and continuous on \([a, b]\). Then there is \( c \in (a, b) \) such that
  \[
  \frac{f(b) - f(a)}{b - a} = f'(c).
  \]
- \( f' > 0 \) on \((a, b) \implies f \) is strictly increasing on \((a, b)\).
- \( f' \geq 0 \) on \((a, b) \implies f \) is increasing on \((a, b)\).
- \( f' \geq 0 \) on \((a, b) \iff f \) is increasing on \((a, b)\).
- \( f(x) = x^3 \) shows that \( f' \geq 0 \) on \((a, b) \iff f \) is strictly increasing on \((a, b)\).
Extreme point 1

- If $c$ is an extreme point and $f'(c)$ exists, then $f''(c) = 0$.
- First Derivative Test: If $f'$ exists around $c$, and $f'$ changes sign at $c$, then $c$ is an extreme point.
- Second Derivative Test: If $f'(c) = 0$ and $f''(c)$ is positive (negative), then $c$ is a minimum (maximum).

Extreme point 2

- If $f'$ changes sign at $c$, then $c$ is an extreme point. The converse is not always true.
- $f(x) = x^2(2 + \sin(1/x))$, $f'(x) = 4x + 2x \sin(1/x) - \cos(1/x)$.
- $x^2 + x^2 \sin(1/x))$ has infinitely many zeros.
- If $f'$ is positive on $(a, b)$, then $f$ is increasing on $(a, b)$. But what if we only know that $f'(c) > 0$? Can we say that $f$ is increasing on an interval around $c$?
- $f(x) = x + 2x^2 \sin(1/x)$, $f'(x) = 1 + 4x \sin(1/x) - 2\cos(1/x)$ is both positive and negative in every neighborhood of 0.
We say that \( c \) is a point of inflection if \( f \) has a tangent line at \( c \) and \( f'' \) changes sign at \( c \). (Some people only require that \( f \) should be continuous at \( c \).)

\( f(x) = x^3 \) has \( f'(0) = 0 \), but 0 is not an extremum, but a point of inflection.

\( f(x) = x^3 + x \) shows that \( f' \) does not have to be 0 at a point of inflection.

\( f(x) = x^{1/3} \) has a point of inflection at 0, has a tangent line at 0, but \( f'(0) \) and \( f''(0) \) do not exist. (Vertical tangent line. Just bend a bit, and you get a point of inflection.)

\[
f(x) = \begin{cases} 
  x^2 & \text{if } x \geq 0, \\
  -x^2 & \text{if } x < 0,
\end{cases}
\]

has a point of inflection at 0, and \( f'(0) \) exists, but \( f''(0) \) does not exist. (First derivatives match, so we get a tangent line, but second derivatives do not match.)
1. If \( c \) is a point of inflection and \( f''(c) \) exists, then \( f''(c) = 0 \).
2. If \( c \) is a point of inflection, then \( c \) is an isolated extremum of \( f' \).
3. If \( c \) is a point of inflection, then the curve lies on different sides of the tangent line at \( c \).

Proof of 3: We use MVT go get \( x_1 \) between \( c \) and \( x \) with
\[
\frac{f(x) - f(c)}{x - c} = f'(x_1),
\]
or
\[
f(x) = f(c) + f'(x_1)(x - c).
\]
We now use MVT again to get \( x_2 \) between \( c \) and \( x_1 \) with
\[
\frac{f'(x_1) - f'(c)}{x_1 - c} = f''(x_2),
\]
or
\[
f'(x_1) = f'(c) + f''(x_2)(x_1 - c).
\]
Combining this, we get
\[
f(x) = f(c) + f'(x_1)(x - c) = f(c) + f'(c)(x - c) + f''(x_2)(x - c)(x_1 - c).
\]
Point of inflection 5

- The tangent line to $f(x)$ at $c$ is $t(x) = f(c) + f'(c)(x - c)$, so the distance between $f$ and the tangent is $f''(x_c)(x - c)(x_1 - c)$.
- Since $(x_1 - c)$ and $(x_1 - c)$ have the same sign, their product is positive. But $f''(x)$ changes sign at $c$, so $f(x)$ will lie on different sides of the tangent at $c$.

Point of inflection 6

- Converse to 1 is false: $f(x) = x^4$ has $f''(0) = 0$, but $f''(x) \geq 0$.
- Converse to 2 is false: $f(x) = x^3 + x^4 \sin(1/x)$ has

$$f'(x) = 3x^2 - x^2 \cos(1/x) + 4x^3 \sin(1/x)$$

$$= x^2(3 - \cos(1/x) + 4x \sin(1/x)) \geq 0$$

in a neighborhood of 0, so 0 is an isolated minimum of $f'(x)$. We have $f''(0) = 0$, but $f''(x) = 6x - \sin(1/x) - 6x \cos(1/x) + 12x^2 \sin(1/x)$ does not change sign.
We need to “integrate” the example $2x^2 + x^2 \sin(1/x)$. Since the derivative of $1/x$ is $-1/x^2$, we try

$$f(x) = x^3 + x^4 \sin(1/x),$$

$$f'(x) = 3x^2 - x^2 \cos(1/x) + 4x^3 \sin(1/x)$$

$$= x^2(3 - \cos(1/x) + 4x \sin(1/x)).$$

The first two terms give us the shape we want, and the last terms is so small that we can ignore it.

Converse to 3 is false:

$f(x) = 2x^3 + x^3 \sin(1/x) = x^3(2 + \sin(1/x))$ lies below the tangent ($y = 0$) on one side and above the tangent on another, but $f''(x) = 12x + 6x \sin(1/x) - 4 \cos(1/x) - (1/x) \sin(1/x)$ does not change sign, since when $x$ is small, the last term will be oscillate wildly.

The cubic terms gives the desired shape of the curve, and since the derivative of $1/x$ is $-1/x^2$, we will get a term of the form $(1/x) \sin(1/x)$ in $f''(x)$, which will make it oscillate wildly.
L'Hôpital’s Rule

Let \( f \) and \( g \) be continuous on an interval containing \( a \), and assume \( f \) and \( g \) are differentiable on this interval with the possible exception of the point \( a \). If \( f(a) = g(a) = 0 \) and \( g'(x) \neq 0 \) for all \( x \neq a \), then

\[
\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to a} \frac{f(x)}{g(x)} = L,
\]

for \( L \in \mathbb{R} \cup \infty \).

Assume \( f \) and \( g \) are differentiable on \( (a, b) \) and that \( g'(x) \neq 0 \) for all \( x \in (a, b) \). If \( \lim_{x \to a} g(x) = \infty \) (or \( -\infty \)), then

\[
\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to a} \frac{f(x)}{g(x)} = L,
\]

for \( L \in \mathbb{R} \cup \infty \).

L'Hôpital’s Rule 2

L'Hôpital does not say that

\[
\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \iff \lim_{x \to a} \frac{f(x)}{g(x)} = L.
\]

If \( f(x) = x + \sin x \) and \( g(x) = x \), then

\[
\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{1 + \cos x}{1}
\]

does not exist, while

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \left( 1 + \frac{\sin x}{x} \right) = 1.
\]