



UiO : **University of Oslo**

Algebra

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Roots of quadratic polynomials

- The roots of a quadratic polynomial $ax^2 + bx + c$ are given by

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

and it is clear that we will get rid of the square roots if we add or multiply the roots.

- We get

$$\begin{aligned} x_1 + x_2 &= \frac{-2b}{2a} = \frac{-b}{a} \quad \text{and} \\ x_1 x_2 &= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}. \end{aligned}$$

Rational roots of polynomial equations

- ▶ If $a_n x^n + a_{n-1} x^{n-1} \dots + a_1 x + a_0 = 0$, where $a_i \in \mathbb{Z}$, has a rational root r/s , where $r, s \in \mathbb{Z}$ are relatively prime, then $r \mid a_0$ and $s \mid a_n$. In particular, if $a_n = 1$, the root must be an integer.
- ▶ Proof: We have

$$a_n(r/s)^n + a_{n-1}(r/s)^{n-1} \dots + a_1 r/s + a_0 = 0 \text{ so}$$
$$a_n r^n + a_{n-1} r^{n-1} s \dots + a_1 r s^{n-1} + a_0 s^n = 0.$$

- ▶ This gives us

$$a_0 s^n = r(-a_n r^{n-1} - a_{n-1} r^{n-2} s \dots - a_1 s^{n-1}) \text{ and}$$
$$a_n r^n = s(-a_{n-1} r^{n-1} \dots - a_1 r s^{n-2} - a_0 s^{n-1}).$$

This shows that r divides $a_0 s^n$, and since r and s are relatively prime, we must have $r \mid a_0$. It follows similarly that $s \mid a_n$.

Geometric background 1

- ▶ If $P = (x_0, y_0)$ is the extremum of the parabola $f(x) = ax^2 + bx + c$, then $x_0 = -b/(2a)$.
- ▶ If we have two real roots, this follows from the quadratic formula, since the extremum is halfway between the two real roots.
- ▶ We can also complete the square to get

$$y = ax^2 + bx + c = a\left(x + b/(2a)\right)^2 + c - b^2/(4a),$$

from which we see that the extremum is

$$P = (-b/(2a), (4ac - b^2)/(4a)).$$

- ▶ Notice that the numerator in the second component is the negative of the discriminant.
- ▶ Alternatively we can use calculus and solve $f'(x) = 2ax + b = 0$.

Geometric background 2

- ▶ We can also write this as

$$y + (b^2 - 4ac)/(4a) = a(x + b/(2a))^2,$$

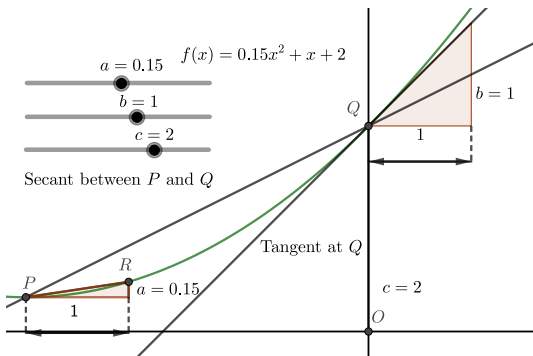
which shows that any parabola can be translated into a parabola of the form $y' = a(x')^2$ by setting $y' = y + (b^2 - 4ac)/(4a)$ and $x' = x + b/(2a)$.

- ▶ It follows that a parabola is determined by the position of the extremum and the value of a , which is related to the curvature at the extremum. (Strictly speaking, the curvature at the extremum is $2a$, but that is not important.)

Geometric background 3

- ▶ There are two important points, namely the extremum $P = (-b/(2a), (4ac - b^2)/(4a))$ and the y -intercept $Q = (0, c)$.
- ▶ We will also consider the secant between the extremum and the y -intercept, and the tangents at the y -intercept and the extremum.
- ▶ $c = f(0)$ is the y -intercept.
- ▶ $b = f'(0)$ is the slope of the tangent at the y -intercept.

Geometric background 4



- ▶ We will show that a is the vertical distance between the point R on the graph that is one unit to the right of the extremum P and the tangent at P .

Geometric background 5

- ▶ This follows from

$$\begin{aligned} & f(-b/(2a) + 1) - f(-b/(2a)) \\ &= a(-b/(2a) + 1)^2 - a(-b/(2a))^2 + \\ & \quad b(-b/(2a) + 1) - b(-b/(2a)) + c - c \\ &= -b + a + b = a. \end{aligned}$$

- ▶ This is similar to the statement that the slope of a straight line is the increase in y if x increases by 1, and helps visualize how a measures the curvature of the parabola.

Geometric background 6

- ▶ We now want to determine the slope of the secant line between the extremum $P = (-b/(2a), (4ac - b^2)/(4a))$ and the y -intercept $Q = (0, c)$.
- ▶ We get

$$\frac{c - (4ac - b^2)/(4a)}{0 - (-b/(2a))} = \frac{b^2/(4a)}{b/(2a)} = \frac{b}{2},$$

so the equation for the secant line is

$$y = -b/(2x) x^2 + bx + c = b/2 x + c.$$

- ▶ Notice that this equation does not contain a . That will be significant later.

Variation of the coefficients 1

- ▶ We are now going to vary the coefficients in $y = ax^2 + bx + c$ and see how the parabola changes. Based on the previous discussion, we will focus on the motion of the extremum by finding the equation for the curve that the path of the extremum traces out.
- ▶ When we vary c , the curvature of the parabola does not change, but the extremum moves up and down along the vertical line $x = -b/(2a)$.
- ▶ Notice that the equation for the curve that the extremum moves on when we vary a coefficient, cannot itself contain that coefficient, as we see with $x = -b/(2a)$ when we vary c .

Variation of the coefficients 2

- ▶ I would like to make a little digression here. What is the equation for a line in the plane?
- ▶ The equation $y = ax + b$ gives you all the non-vertical lines in the plane. If you want to be inclusive and get all the lines, you must write $ax + by = c$.
- ▶ GeoGebra's default option is to use the general formula for lines, but we can set it to write lines as $y = ax + b$ when possible.
- ▶ Mathematicians tend to be inclusive, so why are we so often vertophobic? Are vertical lines different?
- ▶ Vertical lines do not have finite slope, and are not the graph of a function of x .

GeoGebra animation 1

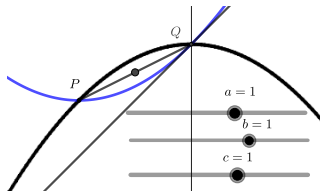
- ▶ We will first do a GeoGebra animation.
- ▶ Enter $f(x) = ax^2 + bx + c$ in GeoGebra and create sliders.
- ▶ Notice that in GeoGebra we can write “ ax ” and “ $=$ ”, while in GeoGebra CAS we have to write “ $a * x$ ” and “ $:=$ ”.
- ▶ Make sure that your students understand that in mathematics, “ $=$ ” is used for both assignment and comparison, while in computer science, we tend to either use “ $=$ / $==$ ” (Python) or “ $:=$ / $=$ ” (GeoGebra CAS).
- ▶ Write $P = \text{Extremum}(f)$ and find the y -intercept Q using the graphical interface.

GeoGebra animation 2

- ▶ We will now vary the coefficients, and study how the graph changes. In order to describe this, we will focus on the behavior of P and Q .
- ▶ Right-click the sliders, select `Object Properties`, and set `Increment` to 0.1.
- ▶ Right-click P and Q , and select `Trace On`.
- ▶ Move the sliders manually or select `Animation On`.
- ▶ The motion of P when we vary c is trivial, and the motion when we vary b is well-known. I thought that I knew what would happen when we vary a , but I was surprised when I first ran the animation.

GeoGebra animation 3

- ▶ Varying b makes the symmetry axis move left and right, in opposite direction to the motion of b , but the graph also moves up and down.
- ▶ It turns out that when we vary b , the extremum moves on the upside down parabola in the picture.



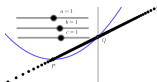
- ▶ We will show later that the upside down parabola is obtained by rotating the original parabola around the midpoint of the secant line between the extremum and the y -intercept.

GeoGebra animation 4

- ▶ To visualize the motion of the extremum when we vary b , we recall that b is the slope of the tangent at the y -intercept. So when $b = 0$, the extremum is at the y -intercept. If $b > 0$ increases while $a > 0$, the parabola must move down and left, in order for the slope at the intercept to have increased, and the extremum must move right and down when $b < 0$ decreases. So the extremum pivots around the y -intercept when b varies.

GeoGebra animation 5

- ▶ I used to think that changing a would just bend the graph up and down, and not move it left and right. In particular, I assumed that the extremum would not move.
- ▶ I was therefore surprised when I actually conducted the experiment and saw that the extremum moved along the dotted line below.



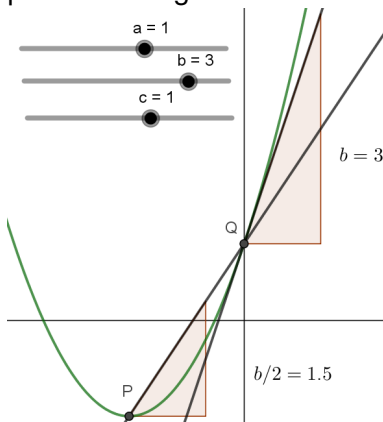
- ▶ When GeoGebra creates sliders, it sets all the coefficients equal to 1, so the default curve is $x^2 + x + 1$.
- ▶ Mathematicians, however, would tend to think of ax^2 as the “mother curve”, and in this case, varying a would just bend the graph up and down, but it would not move left and right.

GeoGebra animation 6

- ▶ The behavior of the graph when a changes depends therefore on whether $b = 0$ or $b \neq 0$. If $b = 0$, we have $x = 0$ and $y = c$, so the extremum will stay fixed at the y -intercept when we vary a , while the curvature and direction of the curve changes.
- ▶ Let us now assume that both a and b are positive. When we vary a , the y -intercept stays fixed and the extremum moves left and right since $x_0 = -b/(2a)$.
- ▶ However, the extremum must also move up and down. When $a > 0$ increases, the curve turns up rapidly, and the extremum moves towards the y -intercept since $x_0 = -b/(2a)$.
- ▶ If $a > 0$ decreases towards 0, the curve becomes flatter, approaches the tangent line $y = bx + c$, and the extremum is pushed towards $-\infty$.
- ▶ The cases when a or b are negative are similar.

GeoGebra animation 7

- ▶ We will show later that the extremum will move on $y = b/2 x + c$, the secant between the extremum and the y -intercept, when we vary a .
- ▶ Notice that this is not the same as $y = bx + c$, the tangent line at the y -intercept, which is also the degenerate parabola we get when we set $a = 0$.



Algebraic analysis 1

- ▶ The equation for the curve that the extremum moves on when we vary a coefficient cannot itself contain that coefficient. Because otherwise the curve would not stay fixed.
- ▶ To find the equation for the curve, we use the equation $x = -b/(2a)$ to eliminate the coefficient in question and find an equation for the curve that involves x , y and the two constant coefficients.
- ▶ In the c -case, the equation $x = -b/(2a)$ does not involve c , so there is nothing to do. The equation for the curve that the extremum moves on is simply $x = -b/(2a)$.

Algebraic analysis 2

- ▶ In the b -case, we have $x = -b/(2a)$, so we solve for b and substitute $b = -2ax$ into the expression for y to get $y = ax^2 + (-2ax)x + c = -ax^2 + c$.
- ▶ Since we eliminated b , we got a curve that is a parabola without a linear term.
- ▶ In the a -case, we solve for a in $x = -b/(2a)$, and substitute $a = -b/(2x)$ into the expression for y to get $y = -b/(2x) x^2 + bx + c = b/2 x + c$.
- ▶ Since we eliminated a , we got a straight line, and we have seen that this line is the secant between the extremum and the y -intercept.

Geometric analysis 1

- ▶ When we vary b , we are looking for a curve V that does not involve b , and which contains the extremum of $y = ax^2 + bx + c$ for all values of b .
- ▶ If we choose an arbitrary $b \neq 0$, then the corresponding parabola U_b will have an extremum P that is different from the y -intercept Q , so that we have a secant between the two points.
- ▶ We rotate the parabola U_b 180° around the midpoint of the secant segment between the extremum and the y -intercept, giving us a parabola V .
- ▶ Since the rotation takes the horizontal tangent at P to a horizontal tangent at Q , it follows that the y -intercept of V is an extremum of V , and it follows that the equation for V does not have a linear term.

Geometric analysis 2

- ▶ Since rotation changes the sign of the quadratic term and the two parabolas have the same y -intercept, it follows that the equation of V is $y = -ax^2 + c$.
- ▶ Notice that we get the same rotated parabola V no matter which U_b with $b \neq 0$ we start with, and that by construction, V contains the extremum of U_b .
- ▶ For U_0 the extremeum is the y -intercept, which is also on V , so we see that V is the path of the extremum when we vary b .

Geometric analysis 3

- ▶ When we vary a with $b \neq 0$, we start with an arbitrary $a \neq 0$ to get a parabola R_a where the extremum is different from the y -intercept. We define S to be the punctured secant obtained by removing the y -intercept from the secant between the extremum and the y -intercept of R_a .
- ▶ We have shown that the equation of S does not contain a , and by definition S contains the extremum, so the path of the extremum is the punctured secant S .
- ▶ When we vary a , we must consider positive and negative values separately, which is why we get a punctured curve. The extremum cannot reach the y -intercept, since if the extremum coincides with the y -intercept, we must have $b = 0$.
- ▶ However, in the $b = 0$ case, the extremum will stay fixed at the y -intercept when we vary a , so our geometric object when we vary a with $b = 0$ is simply the y -intercept, which was removed in the $b \neq 0$ case.