

Possible topics and questions on the exam

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An introduction to convexity

Chap. 1

- What is a convex set? What is an affine set? Examples: the unit cube, hyperplanes.
- Is the sum of convex sets also convex?
- What is a cone? Example: First octant in $\mathbb{R}^3$.
- What is a polyhedron?

Chap. 2

- What is the convex/affine hull of a set? Is the convex hull a convex set?
- What is a polytope? Explain why a polytope is compact? (see p. 32)
- Affine independence and connection with linear independence
- What is the (affine) dimension of a set?
- What is a simplex?
- What is the relative topology? Look at the unit disk of $\mathbb{R}^2$ embedded in $\mathbb{R}^3$. What is the relative interior? The interior? Relative boundary?
- Can all points in a convex set S be expressed in terms of affinely independent points in S ? If so, how many do you need? And do the convex combinations of these generate all of S ? (p. 29)

Chap. 3

- What is a supporting/separating hyperplane?
- Suppose that C is closed and convex, and $\mathbf{x}$ is not in C . Can $\mathbf{x}$ and C be separated with a hyperplane? (See p. 41).
- What does Farkas lemma say? (See p. 44). Can you sum up its proof? (separating hyperplane theorem)

Chap. 4

- What is a face of a convex set? What are the faces of different dimensions for the unit cube of $\mathbb{R}^3$?
- What is an extreme point? What are those for the unit cube and the unit sphere?
- Given a polytope $\text{conv}(x_1, \dots, x_t)$. What can you say about the extreme points?
- Let S be a subset of vectors with all components being either 0 or 1. Explain why the extreme points of $\text{conv}(S)$ are precisely S .
- What is the recession cone of a closed convex set?
- Suppose C is a convex compact set. What does Minkowski's theorem say about C ? ($C = \text{conv}(\text{ext}(C))$)
- What is a vertex of a polyhedron? Connection between vertices and extreme points.
- Why is the intersection of two polytopes again a polytope?
- What does the main theorem for polyhedra say? ($P = \text{conv}(V) + \text{conv}(Z)$)

Chap. 5

- What is a convex function?
- What is the slope function, and its connection to convexity?
- What is the epigraph of a function? What can you say about the epigraph of a convex function?

Chap. 6

- Write down the general form of an optimisation problem with equality- and inequality constraints. What do we mean by regular points and active constraints?
- What do we mean by the KKT conditions? What is the connection with Lagrange's multiplier method?

Notes on combinatorial optimisation

Chap. 1

- State the general combinatorial optimisation problem. What is the node-edge incidence matrix of a graph?
- What is the forest polytope ($F(G)$) of a graph?

- Can you describe $F(G)$ as a polyhedron (Q)? What are the constraints? ($\mathbf{x}(E[S]) \geq |S| - 1$) Say something about the proof (look at the vertices, characterise them as optimal solutions of an LP problem, show by complementary slack that the greedy algorithm finds optimal solution. This is an incidence vector).
- What is a formulation for a set S ?
- What is a Hamilton-tour in a graph? Can you state (in)equalities which characterise incidence vectors of Hamilton cycles? (Subtour/degree constraints, see p.7)
- What is the traveling salesman problem (TSP)?
- What is a relaxation to an LP problem. Why do we consider relaxations for problems such as the TSP?
- What is a separation oracle (p. 12)? Why do we use these?

Chap. 2

- What is an integral polyhedron? What is a rational polyhedron?
- What is a totally unimodular (TU) matrix? Is a matrix with -1 in all entries TU?
- If A is TU and $\mathbf{b}$ is integral, what can you say about the polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$?
- When is the node-edge incidence matrix of a graph TU?
- Explain the following concepts in a graph: matching, node cover, node packing, edge cover.

Chap. 3

Explain Christofides algorithm for finding a Hamilton tour in a weighted graph. What does it mean that this is a factor 2 approximation algorithm?

Chap. 4

- What is an enumeration tree? What do we mean by branch and bound?
- What do we mean by pruning? Can you state a case where pruning can be applied?
- What do we mean by valid inequalities for a set S ?

Network flows and combinatorial matrix theory

Chap. 1

- What is a flow in a graph?
- What is the divergence of a flow?
- What is a circulation in a graph?
- What does Hoffman's circulation theorem say? The proof is rather technical, but you should understand the role of the auxiliary graph therein.
- What is an st-flow, and the value of a flow?
- What is an st-cut?
- What is the Maximum st-flow problem, and what is the minimum st-cut problem? What is the connection between the two?
- How is Hoffman's theorem used in the proof? Algorithm (Ford Fulkerson and x-augmenting paths)

Chap. 2

- What is a doubly stochastic matrix? Is the set of doubly stochastic matrices convex?
- What does it mean that a vector $\mathbf{x}$ is majorised by a vector $\mathbf{y}$?
- State the Gale Ryser theorem, in particular its connection to majorisation and given row/column sums.
- Let R and S be vectors with positive integers. When can we find an integral matrix A with row sums R , column sums S , and so that $0 \leq A \leq C$ (with C integral)? Can this be related to a graph?
- What are the vertices/extreme points for the set of doubly stochastic matrices?