MAT4200 Fall 2012
Mandatory assignment

Due date: Thursday November 1, 2.30 PM

Fill out the cover page for mandatory assignments, staple it to your solution set, and put it in the designated box on the 7th floor of the Niels Henrik Abel building.

Problem 1

Let A be a ring. The *support* of an A-module M is defined as

$$\operatorname{Supp}(M) := \{\mathfrak{p} \subset A \text{ prime ideal }, M_{\mathfrak{p}} \neq 0\}.$$

a) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence. Show that

$$\operatorname{Supp}(M) = \operatorname{Supp}(M') \cup \operatorname{Supp}(M'').$$

If $\mathfrak{a} \subset A$ is an ideal, we write

$$Z(\mathfrak{a}) := \{\mathfrak{p} \subset A \text{ prime ideal} \mid \mathfrak{p} \supseteq \mathfrak{a}\}.$$

b) Show that $Z(\mathfrak{a}) = \operatorname{Supp}(A/\mathfrak{a})$.

c) Show that $\operatorname{Supp}(M) \subseteq Z(\operatorname{Ann} M)$ for any A-module M, and that equality holds if M is finitely generated.

Problem 2

Let $f\colon A \to B$ be a ring homomorphism, M an A-module. Set $M_B := B \otimes_A M$. Assume B is flat as an A-module, and that for every maximal ideal $\mathfrak{m}$ in A, there exists a maximal ideal in B containing $\mathfrak{m}B$. Show that if $M \neq 0$, then $M_B \neq 0$.

Problem 3

Let A be a ring, $\mathfrak{p}$ a prime ideal, $S = A \setminus \mathfrak{p}$. If $\mathfrak{a}$ is an ideal of A, set $S(\mathfrak{a}) := (S^{-1}\mathfrak{a})^{\mathrm{c}}$, in other words, $S(\mathfrak{a}) = \mathfrak{a}^{\mathrm{ec}}$ is the contraction of the extension of the ideal $\mathfrak{a}$ with respect to the homomorphism $A \to S^{-1}A$ that sends x to $\frac{x}{1}$. The nth symbolic power of $\mathfrak{p}$ is defined as $\mathfrak{p}^{(n)} := S(\mathfrak{p}^n)$. Show that

a) $\mathfrak{p}^{(n)}$ is $\mathfrak{p}$-primary

b) $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ if and only if $\mathfrak{p}^n$ is $\mathfrak{p}$-primary.

Problem 4

Consider the ring $A = \mathbb{Z}[X]$ and the ideal $\mathfrak{a} = (9X, 3X^2)$. Find a minimal primary decomposition of $\mathfrak{a}$ and the prime ideals associated with $\mathfrak{a}$. Which of these are minimal?

Problem 5

a) Assume that A and B are integral domains, that A is a subring of B, and that the fields of fractions $K(A)$ and $K(B)$ of A and B are equal. Show that if B is integrally closed and B is integral over A, then B is the integral closure of A.

b) Let k be a field, and set $A = k[x, y, z]/(xy^2 - z^2)$. You can assume (without proving it) that A is an integral domain. Show that the field of fractions of A is equal to

$$K(A) = k(\frac{z}{y}, y),$$

and find the integral closure of A.