MAT4200 Fall 2012

Suggested solutions to the exam problems

Problem 1

a): We have $\varphi_M(x) = \frac{x}{1} = 0$ iff there is an element $s \in S = A \setminus 0$ such that $sx = 0$. This is the same as saying that $\operatorname{Ann}(x) \neq 0$.

b): Consider the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & T(M') & \longrightarrow & T(M) & \longrightarrow & T(M'') \\
\downarrow & & \downarrow{\scriptstyle \varphi_{M'}} & & \downarrow{\scriptstyle \varphi_M} & & \downarrow{\scriptstyle \varphi_{M''}} \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & S^{-1}M'' & \longrightarrow & S^{-1}M & \longrightarrow & S^{-1}M''
\end{array}$$

The two lower rows and the columns are exact, and all the squares are commutative. It follows from (part of) the snake lemma that the upper row is also exact.

Problem 2

a): The polynomial $X^2 + X + 1$ is irreducible in A, so $\mathfrak{a}_1$ is a prime ideal, hence equal to its own radical. The polynomial $X^2 + X + 1$ is not irreducible in $\mathbb{Z}/(3)[X]$: since $1 = -2$ in $\mathbb{Z}/(3)$, $X^2 + X + 1 = X^2 - 2X + 1 = (X-1)^2$. So $r(\mathfrak{a}_1) = r(r(3) + r((X-1)^2)) = r(3, X-1) = (3, X-1)$. The last equality follows since $(3, X-1)$ is maximal, because $A/(3, X-1) \cong \mathbb{Z}/(3)$ is a field. Finally, $r(\mathfrak{a}_3) = (r(4X^3) + r(8X^2)) = r((2X) + (2X)) = r(2X) = (2X)$.

b): The ideal $\mathfrak{a}_1$ is a prime ideal. Since $r(\mathfrak{a}_2) = (3, X-1)$ is a maximal ideal, $\mathfrak{a}_2$ is $(3, X-1)$-primary. By **2a**, $r(\mathfrak{a}_3) = (2X) = (2) \cap (X)$, so is equal to the intersection of two distinct prime ideals. Hence $\mathfrak{a}_3$ is not primary.

c): We claim that $(4X^n, 2^n X^2) = (4) \cap (X^2) \cap (2^n, X^n)$. Clearly both $4X^n$ and $2^n X^2$ are contained in the three ideals to the right. Conversely, assume $f \in 2^n g + X^n h \in (4) \cap (X^2)$. Then we must have $X^2 | g$ and $2^n | h$. Hence $f \in (4X^n, 2^n X^2)$.

The ideal (4) is (2)-primary (since if $fg \in (4)$ and $f \notin (2)$, then $g \in (4)$), (X^2) is (X)-primary (since if $fg \in (X^2)$ and $f \notin (X)$, then $g \in (X^2)$), and $(2^n, X^n)$ is $(2, X)$-primary (since $r(2^n, X^n) = (2, X)$ is maximal). The decomposition is minimal (since $n \geq 3$).

Problem 3

a): An integral domain with Krull dimension 0 is a field. If B is integral over a field, then B is a field. An example is $A = \mathbb{R}$ and $B = \mathbb{C} = \mathbb{R}[i]$. Then $\mathbb{C}$ is integral over $\mathbb{R}$ since $i^2 + 1 = 0$.

An example where A has Krull dimension 1 is $A = \mathbb{Z}$ and $B = \mathbb{Z}[i]$, the Gaussian integers.

b): First of all, note that $B_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}} = S^{-1}B$, where $S = A \setminus \mathfrak{p}$, whereas $B_{\mathfrak{q}} = T^{-1}B$, where $T = B \setminus \mathfrak{q}$. Assume $y \in B \setminus \mathfrak{q}$. Then $\frac{1}{y} \in B_{\mathfrak{q}}$. Assume $\frac{1}{y}$ is integral over $A_{\mathfrak{p}}$. Then there is a relation

$$(\frac{1}{y})^n + \frac{a_1}{s_1}(\frac{1}{y})^{n-1} + \cdots + \frac{a_n}{s_n} = 0,$$

where $a_i \in A$, $s_i \in A \setminus \mathfrak{p}$. Multiplying with y^{n-1} gives

$$\frac{1}{y} = -\frac{a_1}{s_1} - \cdots - \frac{a_n}{s_n} y^{n-1} \in B_{\mathfrak{p}}.$$

c): Since $\mathfrak{q}$ and $\mathfrak{q}'$ are distinct, we may assume there is a $y \in \mathfrak{q}'$, $y \notin \mathfrak{q}$. If $B_{\mathfrak{q}}$ is integral over $A_{\mathfrak{p}}$, then $\frac{1}{y} \in B_{\mathfrak{q}}$ is integral over $A_{\mathfrak{p}}$. By **2b** it follows that $\frac{1}{y} \in B_{\mathfrak{p}}$. Since $y \in \mathfrak{q}'$, we get $\frac{y}{1} \in \mathfrak{q}' B_{\mathfrak{p}}$. But then $\frac{1}{y}\frac{y}{1} = 1 \in \mathfrak{q}' B_{\mathfrak{p}} \subseteq \mathfrak{q}' B_{\mathfrak{q}'}$, since $B_{\mathfrak{p}} \subseteq B_{\mathfrak{q}'}$. Hence 1 is contained in the maximal ideal of the local ring $B_{\mathfrak{q}'}$, which is a contradiction.

Problem 4

a): The map $A \to A$ given by multiplication with f is a ring homomorphism, and also a homomorphism of A-modules. It takes homogeneous elements of degree n to homogeneous elements of degree $n + r$. Hence, if we change the grading and consider the graded A-module $A(-r)$ so that the map becomes $A(-r) \to A$, then $\varphi(A(-r)_n) = \varphi(A_{n-r}) \subseteq A_n$, so it is a homomorphism of graded A-modules. It follows from the exact sequence

$$0 \to A(-r) \to A \to B = A/(f) \to 0$$

that $P(B, t) = P(A, t) - P(A(-r), t) = (1 - t^r)P(A, t)$. We know that $P(A, t) = (1 - t)^{-3}$, so that we get

$$P(B, t) = (1 - t^r)(1 - t)^{-3} = (1 + t + \cdots + t^{r-1})(1 - t)^{-2}.$$

b): We have

$$P(B, t) = (1 + t + t^2)(1 - t)^{-2}.$$

We can write

$$(1 - t)^{-2} = \Sigma_{n \geq 0} \binom{n+1}{1} t^n = \Sigma_{n \geq 0}(n + 1)t^n,$$

so we need to find the coefficient of t^n in the expression

$$\Sigma_{n \geq 0}(n + 1)t^n + t\Sigma_{n \geq 0}(n + 1)t^n + t^2\Sigma_{n \geq 0}(n + 1)t^n.$$

Rewrite as

$$\Sigma_{n \geq 0}(n + 1)t^n + \Sigma_{n \geq 1} nt^n + \Sigma_{n \geq 1}(n - 1)t^n = 1 + \Sigma_{n \geq 1} 3nt^n.$$

So the Hilbert polynomial of B is $h(n) = 3n$.

We know that $h(n) = \dim_k B_n$ for $n \geq \deg f(t)$, where in this case $f(t) = t^2+t+1$ has degree 2. We observe that $\dim_k B_0 = 1 \neq h(0) = 0$, and that $\dim_k B_1 = 3 = h(1)$. So the equality holds for $n \geq 1$.