Noether Normalization

Lemma 1. *Let $f \in k[x_1, \ldots, x_n]$, $n \geq 2$, be a non-zero polynomial over an infinite field k. Then there are elements $\lambda, a_1, \ldots, a_{n-1} \in k$ such that the polynomial*

$$\lambda f(y_1 + a_1 y_n, \ldots, y_{n-1} + a_{n-1} y_n, y_n) \in k[y_1, \ldots, y_n]$$

is monic in y_n.

Proof. Let f_d be the homogenous part of f of highest degree. We have

$$f_d(\lambda x_1, \ldots, \lambda x_n) = \lambda^d f(x_1, \ldots, x_n)$$

where d is the degree of f. Since k is infinite we can always find $a_1, \ldots, a_{n-1}$ such that $f_d(a_1, \ldots, a_{n-1}, 1) \neq 0$. Let

$$y_j = x_j - a_j x_n, \quad j = 1, 2, \ldots, n-1 \quad \text{and} \quad y_n = x_n$$

and $\lambda = f_d(a_1, \ldots, a_{n-1}, 1)^{-1}$. Then

$$\begin{aligned} &\lambda f(y_1 + a_1 y_n, \ldots, y_{n-1} + a_{n-1} y_n, y_n) \\ &\qquad = \lambda f_d(a_1, \ldots, a_{n-1}, 1) y_n^d + \text{terms of lower degree in } y_n \end{aligned}$$

Which is a monic polynomial in y_n. □

Theorem 1 (Noether Normalization)**.** *Let R be a finitely generated algebra over a infinite field k, with generators $x_1, \ldots, x_n \in R$. Then there is an injective k-algebra homomorphism $\phi : k[t_1, \ldots, t_r] \to R$ from a polynomial ring to R, such that R is integral over $k[t_1, \ldots, t_r]$.*

Example 1. *Let $R = k[x_1, x_2]/(x_1 x_2 - 1) \cong k[x, \frac{1}{x}]$. Then R is not integral over $k[x]$. In fact, suppose $\frac{1}{x}$ is integral over $k[x]$. Then there would exist a polynomial equation*

$$(\frac{1}{x})^n + a_1(x)(\frac{1}{x})^{n-1} + \cdots + a_n(x) = 0$$

where $a_i(x) \in k[x]$. Multiplying by x^{n-1} now gives

$$\frac{1}{x} = -a_1(x) - a_2(x)x - \cdots - a_n(x)x^{n-1} \in k[x]$$

and $\frac{1}{x} \in k[x]$, which is not true.

If we introduce new coordinates, $x_1 = t_1 + t_2$, $x_2 = t_2$, we can write $R = k[t_1, t_2]/(t_2^2 + t_1 t_2 - 1)$. Then there is an injective map $\phi : k[t_1] \to R$, and R is integral over $k[t_1]$.

Proof. We prove the statement by induction on the number n of generators of R. If $n = 1$ we let $t_1 = x_1$.

Assume $n > 1$. If the generators $x_1, \ldots, x_n$ are algebraically independent, we choose $t_i = x_i$, and the result follows.

Suppose there is an algebraic dependence between the generators, i.e. a non-zero polynomial f over k such that $f(x_1, \ldots, x_n) = 0$. Let f_d be the homogenous part of the highest degree of f. By lemma 1 we can find $a_1, \ldots, a_{n-1}$ such that

$$\lambda f(y_1 + a_1 y_n, \ldots, y_{n-1} + a_{n-1} y_n, y_n) \in k[y_1, \ldots, y_n]$$

is monic in y_n. The new coordinates are given by

$$y_1 = x_1 - a_1 x_n, \ \ldots, y_{n-1} = x_{n-1} - a_{n-1} x_n, y_n = x_n$$

and it follows that

$$\lambda f(y_1 + a_1 y_n, \ldots, y_{n-1} + a_{n-1} y_n, y_n) = \lambda f(x_1, \ldots, x_n) = 0$$

Notice that the subalgebra $k[y_1, \ldots, y_n]$ of R, generated by $y_1, \ldots, y_n$ is the same as generated by $x_1, \ldots, x_n$, i.e. all of R. Moreover, y_n is integral over the subalgebra

$k[y_1, \ldots, y_{n-1}]$ by the result of lemma 1. By the induction hypothesis there is a injective algebra homomorphism $\phi : k[t_1, \ldots, t_r] \to k[y_1, \ldots, y_{n-1}]$, such that $k[y_1, \ldots, y_{n-1}]$ is integral over $k[t_1, \ldots, t_r]$. But y_n is integral over $k[y_1, \ldots, y_{n-1}]$, and by the Tower Law of integrality, y_n is integral over $k[t_1, \ldots, t_r]$, and the result follows. $\square$

Let us see how the proof works for the above example.

Example 2. *Let $R = k[x_1, x_2]/(x_1x_2 - 1)$. Then $f(x_1, x_2) = x_1x_2 - 1 = 0$, and the homogenous part of highest degree is $f_2(x_1, x_2) = x_1x_2$. We have $\lambda^{-1} = f_2(1, 1) = 1$. Let $y_1 = x_1 - x_2$, $y_2 = x_2$. Then*

$$\lambda f(y_1 + y_2, y_2) = (y_1 + y_2)y_2 - 1 = y_2^2 + y_1y_2 - 1 = x_1x_2 - 1 = 0$$

and y_2 is integral over $k[y_1]$. Let $t_1 = y_1$. Then $k[t_1] \hookrightarrow k[y_1, y_2] = k[x_1, x_2]$ is an integral extension.