Lemma 1. Let $f \in k[x_1, \ldots, x_n]$, $n \geq 2$, be a non-zero polynomial over an infinite field $k$. Then there are elements $\lambda, a_1, \ldots, a_{n-1} \in k$ such that the polynomial

$$\lambda f(y_1 + a_1y_n, \ldots, y_{n-1} + a_{n-1}y_n, y_n) \in k[y_1, \ldots, y_n]$$

is monic in $y_n$.

Proof. Let $f_d$ be the homogenous part of $f$ of highest degree. We have

$$f_d(\lambda x_1, \ldots, \lambda x_n) = \lambda^df(x_1, \ldots, x_n)$$

where $d$ is the degree of $f$. Since $k$ is infinite we can always find $a_1, \ldots, a_{n-1}$ such that $f_d(a_1, \ldots, a_{n-1}, 1) \neq 0$. Let $y_j = x_j - a_jx_n$, $j = 1, 2, \ldots, n-1$ and $y_n = x_n$ and $\lambda = f_d(a_1, \ldots, a_{n-1}, 1)^{-1}$. Then

$$\lambda f(y_1 + a_1y_n, \ldots, y_{n-1} + a_{n-1}y_n)$$

$$= \lambda f_d(a_1, \ldots, a_{n-1})y_n^d + \text{terms of lower degree in } y_n$$

Which is a monic polynomial in $y_n$. \qed

Theorem 1 (Noether Normalization). Let $R$ be a finitely generated algebra over an infinite field $k$, with generators $x_1, \ldots, x_n \in R$. Then there is an injective $k$-algebra homomorphism $\phi : k[t_1, \ldots, t_r] \to R$ from a polynomial ring to $R$, such that $R$ is integral over $k[t_1, \ldots, t_r]$.

Example 1. Let $R = k[x_1, x_2]/(x_1x_2 - 1) \cong k[x, \frac{1}{x}]$. Then $R$ is not integral over $k[x]$. In fact, suppose $\frac{1}{x}$ is integral over $k[x]$. Then there would exist a polynomial equation

$$\left(\frac{1}{x}\right)^n + a_1(x)(\frac{1}{x})^{n-1} + \cdots + a_n(x) = 0$$

where $a_i(x) \in k[x]$. Multiplying by $x^{n-1}$ now gives

$$\frac{1}{x} = -a_1(x) - a_2(x)x - \cdots - a_n(x)x^{n-1} \in k[x]$$

and $\frac{1}{x} \in k[x]$, which is not true.

If we introduce new coordinates, $x_1 = t_1 + t_2$, $x_2 = t_2$, we can write $R = k[t_1, t_2]/(t_2^2 + t_1t_2 - 1)$. Then there is an injective map $\phi : k[t_1] \to R$, and $R$ is integral over $k[t_1]$.

Proof. We prove the statement by induction on the number $n$ of generators of $R$.

If $n = 1$ we let $t_1 = x_1$.

Assume $n > 1$. If the generators $x_1, \ldots, x_n$ are algebraically independent, we choose $t_i = x_i$, and the result follows.

Suppose there is an algebraic dependence between the generators, i.e. a non-zero polynomial $f$ over $k$ such that $f(x_1, \ldots, x_n) = 0$. Let $f_d$ be the homogenous part of the highest degree of $f$. By lemma 1 we can find $a_1, \ldots, a_{n-1}$ such that

$$\lambda f(y_1 + a_1y_n, \ldots, y_{n-1} + a_{n-1}y_n, y_n) \in k[y_1, \ldots, y_n]$$

is monic in $y_n$. The new coordinates are given by

$$y_1 = x_1 - a_1x_n, \ldots, y_{n-1} = x_{n-1} - a_{n-1}x_n, y_n = x_n$$

and it follows that

$$\lambda f(y_1 + a_1y_n, \ldots, y_{n-1} + a_{n-1}y_n, y_n) = \lambda f(x_1, \ldots, x_n) = 0$$

Notice that the subalgebra $k[y_1, \ldots, y_n]$ of $R$, generated by $y_1, \ldots, y_n$, is the same as generated by $x_1, \ldots, x_n$, i.e. all of $R$. Moreover, $y_n$ is integral over the subalgebra
By the induction hypothesis there is a injective algebra homomorphism \( \phi : k[t_1, \ldots, t_r] \to k[y_1, \ldots, y_{n-1}] \), such that \( k[y_1, \ldots, y_{n-1}] \) is integral over \( k[t_1, \ldots, t_r] \). But \( y_n \) is integral over \( k[y_1, \ldots, y_{n-1}] \), and by the Tower Law of integrality, \( y_n \) is integral over \( k[t_1, \ldots, t_r] \), and the result follows.

Let us see how the proof works for the above example.

**Example 2.** Let \( R = k[x_1, x_2]/(x_1x_2 - 1) \). Then \( f(x_1, x_2) = x_1x_2 - 1 = 0 \), and the homogenous part of highest degree is \( f_2(x_1, x_2) = x_1x_2 \). We have \( \lambda^{-1} = f_2(1, 1) = 1 \). Let \( y_1 = x_1 - x_2, y_2 = x_2 \). Then

\[
\lambda f(y_1 + y_2, y_2) = (y_1 + y_2)y_2 - 1 = y_2^2 + y_1y_2 - 1 = x_1x_2 - 1 = 0
\]

and \( y_2 \) is integral over \( k[y_1] \). Let \( t_1 = y_1 \). Then \( k[t_1] \to k[y_1, y_2] = k[x_1, x_2] \) is an integral extension.