UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT4200 — Commutative algebra

Day of examination: Monday 5 December 2016

Examination hours: 14:30 – 18:30

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

If p is a prime number, we denote by $\mathbb{F}_p := \mathbb{Z}/(p)$ the finite field with p elements.

1a

Show that the ring $\mathbb{F}_2[X]/(X^3+X+1)$ is a field, but that $\mathbb{F}_3[X]/(X^3+X+1)$ is not.

1b

Consider the ideal $\mathfrak{p} := (X^3+X+1) \subset \mathbb{F}_2[X]$. Explain why the localized ring $\mathbb{F}_2[X]_{\mathfrak{p}}$ is a discrete valuation ring. Find an element in the field of rational functions $\mathbb{F}_2(X)$ that has valuation equal to -1.

1c

Write $\mathbb{F}_3[X]/(X^3+X+1)$ as a product of local Artinian rings. (Hint: Factor X^3+X+1 in $\mathbb{F}_3[X]$.)

Problem 2

Consider the graded polynomial ring $A := k[x_0, x_1, x_2, x_3]$ where k is a field. Recall that the Hilbert polynomial of A is equal to $h_A(n) = \binom{n+3}{3}$.

2a

Set $M := A/(x_1x_3 - x_2^2)$. Then M is a graded A-module. Explain why its Hilbert polynomial is equal to $h_M(n) = \binom{n+3}{3} - \binom{n+1}{3}$. For which n does $\dim_k M_n = h_M(n)$ hold?

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2b

Set $N := A/(x_1x_3 - x_2^2, x_0x_2 - x_1^2)$. Find the Hilbert polynomial of this graded A-module.

Problem 3

Let k be a field and set $A := k[x, y, z]$. Consider the ideals $\mathfrak{a} := (xz, yz, z^2, x^3)$ and $\mathfrak{b} := (x^3, z)$.

3a

Show that $\mathfrak{b}$ is a primary ideal, and find its radical.

3b

Show that $(x, y, z^2) \cap \mathfrak{b}$ is a minimal primary decomposition of $\mathfrak{a}$. Find the prime ideals belonging to $\mathfrak{a}$. Which is minimal and which is embedded? Can you find a different minimal primary decomposition of $\mathfrak{a}$?

Problem 4

Let A be a ring, B an A-algebra, and M a B-module. The A-derivations from B to M is the set

$$\mathrm{Der}_A(B, M) := \{D \in \mathrm{Hom}_A(B, M) | D(bb') = bD(b') + b'D(b), \forall b, b' \in B\}.$$

4a

Let $\varphi : C \to B$ be a homomorphism of A-algebras. Recall that φ defines, by restriction of scalars, a C-module structure on M; denote this C-module by $M_{[\varphi]}$. Show that there is a natural homomorphism of A-modules

$$\Phi : \mathrm{Der}_A(B, M) \to \mathrm{Der}_A(C, M_{[\varphi]}).$$

4b

Show that Φ is injective if φ is surjective. Explain why $\mathrm{Der}_A(B, M)$ is a B-module.

4c

Assume $A = k$ is a field and that $B = M = k[x, y]$ is the polynomial ring in two variables with coefficients in k. Find two elements of $\mathrm{Der}_k(k[x, y], k[x, y])$ that generate it as a $k[x, y]$-module.

THE END