Exercises, MAT 4200

Exercise 1
Let $R = \mathbb{Z}[i\sqrt{5}] = \{a + ib\sqrt{5} \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$.

a) Show that the map
$$\mathbb{Z}[X]/(X^2 + 5) \simeq \mathbb{Z}[i\sqrt{5}]$$
given by $X \mapsto i\sqrt{5}$ is an isomorphism of rings.

b) Show that $2 \cdot 3 = (1 + i\sqrt{5})(1 - i\sqrt{5})$ and that the four numbers involved are irreducible in $\mathbb{Z}[i\sqrt{5}]$. Remember that an element in a ring is irreducible if it cannot be written as a product of two non-units. It follows that the ring $\mathbb{Z}[i\sqrt{5}]$ is not a UFD.

c) Show that $\mathfrak{a} = (2, 1 + i\sqrt{5}) \subset \mathbb{Z}[i\sqrt{5}]$ is a maximal ideal, and show that $\mathfrak{a}^2 = (2)$. Compute the radical $r(\mathfrak{a}^2)$.

d) Show that the ring R is normal.

Exercise 2
Recall Nakayamas Lemma: If M is a finitely-generated module over a ring R such that $J \cdot M = M$, where J is the Jcaobson radical of R, then $M = 0$.

a) Let K be a field and $A \subset K$ a non-trivial local ring. Show that K can not be a finitely generated A-modul.

b) Let k be a field and $R = k[x, y]$. Let $I = (x, y)^n$ for some $n \geq 1$. Let $f : I \to I$ be a surjective R-linear endomorphism. Show that f is an isomorphism.

c) Let S be a multiplicatively closed subset of the ring R and let M be a finitely generated R-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

d) Let $\mathfrak{a}$ be an ideal of R, and let $S = 1 + \mathfrak{a}$. Show that S is multiplicatively closed and that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}R$.

e) Use exercises c) and d) to prove the following statement: If M is a finitely generated R-module and $\mathfrak{a}$ an ideal of R such that $\mathfrak{a}M = M$, then there exists $x \equiv 1 \pmod{\mathfrak{a}}$ such that $xM = 0$.

Exercise 3
Let $n \in \mathbb{Z}$ be an integer and let S be the multiplicatively closed set $S = \{m \in \mathbb{Z} \mid (m, n) = 1\}$. We denote the ring $S^{-1}\mathbb{Z}$ by $\mathbb{Z}_{(n)}$. If p is a prime, then $\mathbb{Z}_{(p)}$ is the ring $\mathbb{Z}$ localised at the prime ideal (p). This is a local ring with maximal ideal (p).

a) Let $n = 6$, and show that $\mathbb{Z}_{(6)}$ has two maximal ideals, namely $\mathfrak{m}_1 = (3)\mathbb{Z}_{(6)}$ and $\mathfrak{m}_2 = (2)\mathbb{Z}_{(6)}$.

b) Show that $\mathbb{Z}_{(6)}/\mathfrak{m}_1 \simeq \mathbb{F}_3$ and that $\mathbb{Z}_{(6)}/\mathfrak{m}_2 \simeq \mathbb{F}_2$. Remember that $\mathbb{F}_p$ is the unique field with p elements.

c) What is the Jacobson radical $\mathcal{R}$ of $\mathbb{Z}_{(6)}$? Describe $\mathbb{Z}_{(6)}/\mathcal{R}$.

d) In general, for any $n \in \mathbf{Z}$, show that $\mathbb{Z}_{(n)}$ is a semilocal ring, i.e. a ring with only finitely many ideals. Describe the maximal ideals ant their residue fields.

Exercise 4

a) Show that $\mathbb{Z}/(m) \otimes_{\mathbb{Z}} \mathbb{Z}/(n) \cong \mathbb{Z}/(d)$, where d is the greatest common divisor of m and n. In particular $\mathbb{Z}/(m) \otimes_{\mathbb{Z}} \mathbb{Z}/(n) = 0$ if m and n are relative prime.

b) Prove that if M and N are flat R-modules, then $M \otimes_A N$ is a flat R-module.

c) Let V be a finite dimensional vector space over a field k. The dual V^* of V is the vector space of linear functionals on V with pointwise addition and multiplication. Consider the natural map
$$\phi : V^* \otimes_k V \to End_k(V)$$
given by $\phi(\alpha \otimes v)(w) = \alpha(v)w$, where $\alpha \in V^*$, and $v, w \in V$. Show that ϕ is an isomorphism.

d) Let $\{e_1, \ldots, e_n\}$ be a basis for a vector space V, and $\{f_1, \ldots, f_m\}$ a basis for a vector space W. Write up a basis the vector space $V \otimes_k W$. What is its dimension over k?

Exercise 5

a) Let $x \in R$ be a nilpotent element. Show that $1 + x$ is a unit in R.

b) Let $R = k[x, y]/\mathfrak{a}$, where k is a field, and $\mathfrak{a} = (xy)$. Describe the set $\mathrm{Spec}(R)$ of prime ideals of R, and give an geometrical interpretation of the set of maximal elements.

c) Let R be as in b) and let $\psi : k[x] \to R$ be the obvious morphism. Let $\mathfrak{p}$ be a non-trivial prime ideal of $k[x]$. The fiber at $\mathfrak{p}$ of the induced map $Spec(R) \to Spec(k[x])$ is the set of prime ideals $\mathfrak{q} \subset R$ such that $\psi^{-1}(\mathfrak{q}) = \mathfrak{p}$. Describe the fiber at $\mathfrak{p} = (x - a)$ in the two cases $a = 0$ and $a \neq 0$.

d) Let k be a field, and set $R = k[X, Y]/\mathfrak{b}$, where $\mathfrak{b} = (Y^2 - X^2 - X^3)$. Let x, y be the residues of X, Y in R. Prove that R is a domain, but not a field. Set $t = \frac{x}{y} \in \mathrm{Frac}(R)$. Prove that $k[t]$ is the integral closure of R in $\mathrm{Frac}(R)$.