# MAT 4200 Commutative algebra

## Part 3, Chain conditions, Noetherian rings and modules (Chapt. 9)

October, 1, - October, 8, 2020

# October, 1, 2020

We define Noetherian and Artinian modules and rings and look at some basic properties.

9.1-9.2

### Definition

*An A-module M is said to be* **finitely generated** *if there exists elements* $m_1, \ldots, m_r$ *in M such that any element* $m \in M$ *can be written as a linear combination*

$$m = a_1 m_1 + \ldots a_r m_r$$

*of the* $m_i$*'s with coefficients* $a_i$ *in A.*

Notice that the coefficients $a_i$ are not unique in general; there might be non-trivial relations between the $m_i$'s, and for most modules this occurs for any set generators. The set of generators $\{m_i\}$ gives rise to a surjective map

$$A^r \to M \to 0$$

which sends $(a_i)$ to $\sum_i a_i m_i$.

Given a ring $A$ and an $A$-module $M$.

Definition

*An* **ascending chain** *of submodules is a (countable) chain of submodules $M_i$ shaped like*

$$M_0 \subseteq M_1 \subseteq \ldots . \subseteq M_i \subseteq M_{i+1} \subseteq \ldots,$$

*or phrased differently, a totally ordered countable subset of $\mathscr{I}(M)$.*

*In similar fashion, a* **descending chain** *is one shaped like*

$$\ldots \subseteq M_{i+1} \subseteq M_i \subseteq \ldots \subseteq M_1 \subseteq M_0.$$

### Definition

*A chain is said to be* **eventually constant** *if there is an index $i_0$ so that the members of the chain are equal from $i_0$ and on; that is, $M_i = M_j$ for $i, j \geq i_0$. Common usage is also to say the chain* **stabilizes** *at $i_0$.*

$$M_0 \subseteq M_1 \subseteq \ldots . \subseteq M_{i_0} = M_{i_0+1} = M_{i_0+2} = \ldots,$$

## Example

*Given an ideal* $\mathfrak{a}$ *in the ring* $A$ *the powers* $\mathfrak{a}^i$ *of* $\mathfrak{a}$ *form a descending chain*

$$\ldots \subseteq \mathfrak{a}^{i+1} \subseteq \mathfrak{a}^i \subseteq \ldots \subseteq \mathfrak{a}^2 \subseteq \mathfrak{a}$$

*For an* $A$*-module* $M$ *we have a descending chain of submodules;*

$$\ldots \subseteq \mathfrak{a}^{i+1} M \subseteq \mathfrak{a}^i M \subseteq \ldots \subseteq \mathfrak{a}^2 M \subseteq \mathfrak{a} M$$

### Example

*Concider the abeiain group $\mathbb{Q}$ (which is a $\mathbb{Z}$-module). For every natural number $i$ one has the submodule of $\mathbb{Q}$ generated by $p^{-i}$; that is, the subgroup $\mathbb{Z} \cdot p^{-i}$ whose elements are of the form $np^{-i}$. Then there is an ascending chain*

$$\mathbb{Z} \subset \mathbb{Z} \cdot p^{-1} \subset \ldots \subset \mathbb{Z} \cdot p^{-i} \subset \mathbb{Z} \cdot p^{-(i+1)} \subset \ldots$$

**Definition**

*The A-module M is* **Noetherian** *if every ascending chain of submodules is eventually constant.*

**Definition**

*The A-module M is* **Artinian** *if every descending chain is eventually constant.*

Definition

i) *A ring $A$ is called* **Noetherian** *if it is a Noetherian as module over itself.*

ii) *It is called* **Artinian** *if it is an Artinian as module over itself.*

The submodules of $A$ are precisely the ideals, so $A$ being Noetherian amounts to ideals of $A$ satisfying the ACC, and similarly, $A$ is Artinian precisely when the ideals comply with the DCC.

**Proposition (Main Theorem of Noetherian modules)**

*Let $A$ be a ring and let $M$ be a module over $A$. The following three statements are equivalent.*

i) *$M$ is Noetherian.*

ii) *Every non-empty set of submodules of $M$ has a maximal member;*

iii) *Every submodule of $M$ is finitely generated over $A$;*

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*Proof.*

i) $\Rightarrow$ ii) Assume that $M$ is Noetherian and let $\Sigma$ be a non-empty set of submodules. Assuming there is no maximal elements in $\Sigma$, one proves easily, by induction on the length, that every finite chain in $\Sigma$ can be strictly extended upwards, which is not the case since $M$ is Noetherian.

ii)$\Rightarrow$ iii) Suppose that every non-empty set of submodules has a maximal element, and let $N$ be a submodule. We shall see that $N$ is finitely generated. Let $\Sigma$ be the set of submodules of $N$ finitely generated over $A$. It is clearly non-empty ($Am \subseteq N$ for any $m \in N$) and hence has a maximal element $N_0$. Let $m \in N$ be any element. The module $Am + N_0$ is finitely generated and contained in $N$, and by the maximality of $N_0$, it follows that $m \in N_0$. Hence $N = N_0$, and $N$ is finitely generated.

□

*Proof.* [continues]

iii)⇒ i) Assume that all submodules of $M$ are finitely generated, and let

$$M_0 \subseteq M_1 \subseteq \ldots . \subseteq M_i \subseteq M_{i+1} \subseteq \ldots$$

be an ascending chain. The union $N = \bigcup_i M_i$ is finitely generated, say by $m_1, \ldots, m_r$. Each $m_j$ lies in some $M_{\nu_j}$ and hence they all lies in $M_\nu$ with $\nu = \max_j \nu_j$. Therefore $N = M_\nu$, and the chain is constant above $\nu$.

□

## Proposition

*Let $M'$, $M$ and $M''$ be A-modules that fit in the short exact sequence*

$$0 \to M' \longrightarrow M \xrightarrow{\beta} M'' \to 0$$

*Then $M$ is Noetherian (respectively Artinian) if and only if $M'$ and $M''$ are.*

Remember that

$$\begin{aligned} \phi(\phi^{-1}(x)) = x &\quad \text{if } \phi \text{ is surjective} \\ \phi^{-1}(\phi(x)) = x &\quad \text{if } \phi \text{ is injective} \end{aligned}$$

*Proof.*

- $\Rightarrow$ Every chain in $M'$ is a chain in $M$, and if $M$ is Noetherian, so is $M'$. A chain in $M''$ can be lifted to $M$ and surjectivity of $\beta$ implies $\beta(\beta^{-1}(N)) = N$ for any submodule $N \subseteq M''$. It follows that $M''$ is Noetherian as well.
- $\Leftarrow$ Assume that the two extreme modules $M'$ and $M''$ are Noetherian and let $\{M_i\}_i$ be a chain in $M$. The chain $\{M_i \cap M'\}$ is eventually constant, hence $M_i \cap M' = M_j \cap M'$ for $i, j >> 0$.
  Mapping the $M_i$'s into $M''$ gives the chain $\{\beta(M_i)\}_i$ in $M''$ and since $M''$ is Noetherian by assumption, it is eventually constant. Hence $\beta(M_i) = \beta(M_j)$ for $i, j >> 0$. This gives $M_i / M_i \cap M' = M_j / M_j \cap M'$ and hence $M_i = M_j$.

□

## Proposition

*Let $S$ be a multiplicative set in the ring $A$ and let $M$ be an $A$-module. If $M$ is Noetherian, respectively Artinian, the localized module $M_S$ is Noetherian, respectively Artinian, as well.*

*Proof.*

1. Let $N \subseteq S^{-1}M$. Observ that $S^{-1}(\iota^{-1}(N)) = N$. In fact, since $\frac{m}{1} \in N$ is equivalent to $\frac{m}{s} \in N$ for any $s \in S$, we have
$$\begin{aligned} S^{-1}(\iota^{-1}(N)) &= S^{-1}\{m \in M \mid \tfrac{m}{1} \in N\} \\ &= \{\tfrac{m}{s} \mid \tfrac{m}{1} \in N\} = N \end{aligned}$$
2. Any chain $\{N_i\}$ in $S^{-1}M$, whether ascending or descending, induces a chain $\{\iota^{-1}(N_i)\}$ in $M$, and if this chain stabilizes, say $\iota^{-1}(N_i) = \iota^{-1}(N_j)$ for $i, j \geq i_0$, it holds true that $N_i = S^{-1}(\iota^{-1}N_i) = S^{-1}(\iota^{-1}N_j) = N_j$, and the original chain stabilizes at $i_0$ as well.

□

### Example (Vector spaces)

*A vector space $V$ over a field $k$ is Noetherian if and only if it is of finite dimension.*

*Indeed, if $V$ is of finite dimension it is the direct sum of finitely many copies of $k$, hence Noetherian.*
*If $V$ is not of finite dimension one may find an infinite set $v_1, \ldots, v_i, \ldots$ of linearly independent vectors, and the subspaces $V_i = < v_1, \ldots, v_i >$ will form a strictly ascending chain of subspaces; hence $V$ is not Noetherian. A similar argument shows that neither is $V$ Artinian: The spaces $W_i = < v_i, v_{i+1} \cdots >$ form a strictly decreasing chain of subspaces.*

**Proposition (Main Theorem of Noetherian rings)**

*Let $A$ be a ring. The following three statements are equivalent.*

i) *$A$ is Noetherian.*

ii) *Every non-empty family of ideals in $A$ has a maximal element;*

iii) *Every ideal in $A$ is finitely generated.*

### Proposition

*Let $A$ be a noetherian ring and $M$ an $A$-module. Then $M$ is Noetherian if and only if $M$ is finitely generated.*

*Proof.* A finitely generated $A$-module $M$ can be realized as the quotient of a finite direct sum $nA$ of $n$ copies of $A$. When $A$ is Noetherian, it follows that $nA$ is Noetherian; indeed, one obtains $nA$ by successive extensions of $A$ by itself. All quotients of $nA$, in particular $M$, will be Noetherian. □

## Example

*An example of a non-noetherian ring is the ring $A[x_1, x_2, \ldots]$ of polynomials in infinitely many variables over any ring $A$. The chain of ideals*

$$(x_1) \subset (x_1, x_2) \subset \ldots \subset (x_1, x_2, \ldots, x_i) \subset \ldots$$

*does obviously not stabilize.*

## Example

*One might be mislead by the previous example to belive that non-noetherian rings are monstrously big. There are, however, non-noetherian rings contained in the polynomial ring $\mathbb{Q}[x]$. The simplest example is even a subring of $\mathbb{Z}[p^{-1}][x]$ where $p$ is a natural number greater than one; it is formed by those polynomials in $\mathbb{Z}[p^{-1}][x]$ taking an integral value at zero. In this ring $A$ one has the following ascending chain of ideals*

$$(p^{-1}x) \subset (p^{-2}x) \subset \ldots \subset (p^{-i}x) \subset \ldots$$

*which does not stabilize. Indeed, if $p^{-(i+1)} \in (xp^{-1})$, one has $xp^{-(i+1)} = P(x)xp^{-i}$ for some polynomials $P(x) \in A$. Cancelling $xp^{-i}$ gives us $p^{-1} = P(x)$, and this is a contradiction since $P(0) \in \mathbb{Z}$.*

## Proposition

*Assume that $M$ is a module over $A$. If $M$ is Noetherian, then $A/\mathrm{Ann}(M)$ is Noetherian as well.*

*Proof.* Let $x_1, \ldots, x_r$ be generators for $M$, and consider the map $\phi\colon A \to rM$ that sends $x$ to the tuple $(x \cdot x_1, \ldots, x \cdot x_r)$. If $x$ kills all the $x_i$'s, it kills the entire module $M$, the $x_i$'s being generators, and we may infer that the kernel of $\phi$ equals the annihilator $\mathrm{Ann}(M)$. This means that $A/\mathrm{Ann}(M)$ is isomorphic to a submodule of $rM$, hence it is Noetherian by the proposition above. □

Notice that for a general $A$ we have

| | | | | |
|---|---|---|---|---|
| $M$ Noetherian | $\iff$ | $N \subseteq M$ Finitely generated | $\implies$ | $M$ Finitely generated |

But if $A$ is Noetherian we have

| | | | | |
|---|---|---|---|---|
| $M$ Noetherian | $\iff$ | $N \subseteq M$ Finitely generated | $\iff$ | $M$ Finitely generated |

# October, 7, 2020

The topics today are minimal primes, Hilbert‘s basis theorem and Krull‘s intersection theorem.

9.2-9.3

**Definition**

*The* **minimal prime ideals** *of an ideal* $\mathfrak{a}$ *are the prime ideal minimal among those containing* $\mathfrak{a}$*.*

Remember that for the minimal primes of $\mathfrak{a}$ we have

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subset \mathfrak{p}} \mathfrak{p}$$

The minimal primes are unambiguously determined by $\sqrt{\mathfrak{a}}$, and no inclusion among them can persist.

## Proposition

*Each ideal $\mathfrak{a}$ in a Noetherian ring has only finitely many minimal prime ideals (prime ideals containing $\mathfrak{a}$).*

*Proof.* Let $\Sigma$ be the set of ideals in $A$ with infinitely many minimal prime ideals. If $\Sigma$ is non-empty, it has maximal member, say $\mathfrak{a}$. Obviously $\mathfrak{a}$ is not a prime ideal, so there are elements $x$ and $y$ neither lying in $\mathfrak{a}$, but whose product $xy$ belongs to $\mathfrak{a}$. Then $\mathfrak{a}+(x)$ and $\mathfrak{a}+(y)$ are proper ideals strictly larger $\mathfrak{a}$, and consequently each has merely finitely many minimal primes. Any prime ideal containing $\mathfrak{a}$ contains either $x$ or $y$, hence each minimal prime ideal of $\mathfrak{a}$ is either among the finitely many minimal primes of $\mathfrak{a}+(x)$ or the finitely many of $\mathfrak{a}+(y)$. □

### Corollary

*A radical ideal in a Noetherian ring is an irredundant intersection of finitely many prime ideals. The involved prime ideals are unique.*

*Proof.* A radical ideal is the intersection of the minimal prime ideals containing the ideal. By the proposition there are only finitely many. We also know that if two families of prime ideals have the same intersection, and there are no non-trivial inclusion relations in either family, then the two families coincide. □

### Proposition

*Assume that $A$ is a Noetherian ring and $M$ an $A$-module. Let $Ann(x)$ be maximal among the annihilators of non-zero elements in $M$. Then $Ann(x)$ is a prime ideal.*

*Proof.* To begin with, observe that $\mathrm{Ann}(x)$ is a proper ideal as $x$ is non-zero. Let then $a$ and $b$ be ring elements such that $ab \in \mathrm{Ann}(x)$ and assume that $a \notin \mathrm{Ann}(x)$. Then $ax \neq 0$. It is generally true that $\mathrm{Ann}(x) \subset \mathrm{Ann}(ax)$, but since $ax \neq 0$ it holds that $\mathrm{Ann}(x) = \mathrm{Ann}(ax)$ because $\mathrm{Ann}(x)$ is maximal among annihilators of non-zero elements. Now, $bax = 0$, so $b \in \mathrm{Ann}(ax) = \mathrm{Ann}(x)$. □

## Corollary

*Any non-zero module over a Noetherian ring contains a module isomorphic to $A/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$.*

*Proof.* The set of annihilators of non-zero elements $x \in M$ is non-empty and has a maximal element $\mathrm{Ann}(x)$ since $A$ is Noetherian. By the proposition above this annihilator $\mathrm{Ann}(x)$ is a prime ideal, denote by $\mathfrak{p} = \mathrm{Ann}(x)$. Now consider the homomorphism

$$A \xrightarrow{\cdot x} M$$

The kernel of this map is $\mathrm{Ann}(x)$, thus dividing out by this ideal we get the inclusion in the corollary. □

### Theorem

*Let $A$ a Noetherian ring and let $M$ be a non-zero $A$-module. Then $M$ is finitely generated if and only if it possesses a finite ascending chain of submodules $\{M_i\}_{0\leq i\leq r}$ with $M_0 = 0$ and $M_r = M$ whose subquotients are shaped like cyclic modules $A/\mathfrak{p}_i$ with the $\mathfrak{p}_i$'s being prime; that is, there are short exact sequences*

$$0 \to M_{i-1} \longrightarrow M_i \longrightarrow A/\mathfrak{p}_i \to 0$$

*for $1 \leq i \leq r$.*

*Proof.* Let $M$ be finitely generated $A$-module. The set of submodules of $M$ for which the theorem is true is non-empty by the above corollary and has thus a maximal element, say $N$. If $N$ were a proper submodule, the quotient $M/N$ would be non-zero and hence contain a submodule isomorphic to $A/\mathfrak{p}$ for some prime $\mathfrak{p}$. The inverse image $N'$ of $A/\mathfrak{p}$ in $M$ would be a submodule containing $N$ and satisfying $N'/N \simeq A/\mathfrak{p}$, so the theorem would also hold for $N'$ violating the maximality of $N$. □

$$\begin{array}{ccccc}
N & \longrightarrow & M & \xrightarrow{\ \beta\ } & M/N \\
{\scriptstyle =}\uparrow & & \uparrow & & \uparrow \\
N & \longrightarrow & \beta^{-1}(A/\mathfrak{a}) & \longrightarrow & A/\mathfrak{a}
\end{array}$$

Notice that $N = \beta^{-1}(0)$.

**Theorem (Hilbert's Basis Theorem)**

*Assume that $A$ is a Noetherian ring. Then the polynomial ring $A[x]$ is Noetherian.*

Let

$$f = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in A[x]$$

The element $a_n$ is called the **leading coefficient** of $f$.

*Proof.*

1. Let $\mathfrak{a}$ be an ideal in $A[x]$ and for each $n$ let $\mathfrak{a}_n$ be the set of leading coefficients of elements from $\mathfrak{a}$ of degree at most $n$. Each $\mathfrak{a}_n$ is an ideal in $A$, and they form an ascending chain which eventually stabilizes, say for $n = N$.
2. Each $\mathfrak{a}_n$ is finitely generated, so for each $n \leq N$ we may chose a finite set of polynomials of degree at most $n$ whose leading coefficients generate $\mathfrak{a}_n$. Let $f_1, \ldots, f_r$ be all these polynomials in some order and let $a_1, \ldots, a_r$ be their leading coefficients.
3. We contend that the $f_i$'s generate $\mathfrak{a}$. So assume not, and let $f$ be of minimal degree $n$ among those polynomials in $\mathfrak{a}$ that do not belong to the ideal generated by the $f_i$'s. If the leading coefficient of $f$ is $a$, it holds that $a \in \mathfrak{a}_n$ and we may write $a = \sum_j b_j a_{i_j}$ with the $f_{i_j}$'s corresponding to $a_{i_j}$ of degree at most the degree of $f$. Then $f - \sum_j b_j x^{(\deg f - \deg f_{i_j})} f_{i_j}$ is of degree less than $\deg f$ and does not lie in the ideal generated by the $f_i$'s since $f$ does not, contradicting the minimality of $\deg f$.

## Corollary

- *The polynomial ring $A[x_1, \ldots, x_n]$ over a Noetherian ring $A$ is Noetherian.*
- *Any algebra finitely generated over a Noetherian ring is Noetherian.*
- *Any ring essential of finite type over a Noetherian ring is Noetherian.*

In order to approach Krull's intersection theorem without going into too many technicalities, we state the next result without proof.

## Lemma

*Let $\mathfrak{a} \subset A$ be a finitely generated ideal. Let $M$ be a Noetherian $A$-module and $N$ a submodule. If $K$ is submodule of $M$ maximal subjected to the condition $K \cap N = \mathfrak{a}N$, then $\mathfrak{a}^\nu M \subset K$ for some $\nu \in \mathbb{N}$.*

## Proposition

*Suppose that $A$ is a ring, that $\mathfrak{a}$ is a finitely generated ideal in $A$ and that $M$ is a Noetherian module over $A$. Putting $N = \bigcap_i \mathfrak{a}^i M$, one has $\mathfrak{a}N = N$.*

*Proof.* Because $M$ is Noetherian, there is a maximal submodule $K$ of $M$ such that $K \cap N = \mathfrak{a}N$. By the lemma it holds that $\mathfrak{a}^\nu M \subset K$ for some natural number $\nu$. Because $N \subset \mathfrak{a}^\nu M$, i holds that $N \subset K$, and consequently $N = N \cap K = \mathfrak{a}N$. □

Theorem (Krull's intersection theorem)

*Let $A$ be ring and $\mathfrak{a}$ an ideal contained in the Jacobson radical of $A$. Assume that $\mathfrak{a}$ is finitely generated. If $M$ is a Noetherian $A$-module, it holds true that $\bigcap_i \mathfrak{a}^i M = 0$.*

*Proof.* Let $N = \bigcap_i \mathfrak{a}^i M$. Then $\mathfrak{a}N = N$ by the proposition, and we may finish by applying Nakayama's lemma since $N$ is a submodule of the Noetherian module $M$ and therefore is finitely generated. □

### Corollary

*Let $A$ be a Noetherian ring and $\mathfrak{a}$ an ideal contained in the Jacobson radical of $A$. Then $\bigcap_i \mathfrak{a}^i = 0$. In particular, if $A$ is a Noetherian local ring whose maximal ideal is $\mathfrak{m}$, one has $\bigcap_i \mathfrak{m}^i = 0$.*

## Proposition

*Assume that $\mathfrak{a}$ is a proper ideal in the Noetherian integral domain $A$, then $\bigcap_i \mathfrak{a}^i = 0$.*

*Proof.* Let $N = \bigcap_i \mathfrak{a}^i$. Then $\mathfrak{a}N = N$ by the above proposition. Let $S = 1 + \mathfrak{a}$. Then $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$ because

$$1 + \frac{a}{1+b}\frac{x}{1+y} = \frac{(1+b)(1+y)+ax}{(1+b)(1+y)} \in \frac{1+\mathfrak{a}}{1+\mathfrak{a}}$$

and therefore invertible. Here $a, b, y \in \mathfrak{a}$ and $x \in A$. Now $\mathfrak{a}N = N$ gives $S^{-1}(\mathfrak{a}N) = S^{-1}\mathfrak{a}S^{-1}N = S^{-1}N$ and by Nakayama classic it follows that $S^{-1}N = 0$. By the Noetherian property we have $N = \langle x_1, \ldots, x_r \rangle$ and there are elements $s_i \in S$, $i = 1, 2, \ldots, r$ such that $s_i x_i = 0$. Let $s = \prod_i s_i$. Then we have

$$S = \prod_i s_i = \prod_i (1 + a_i) = 1 + a \quad a \in \mathfrak{a}$$

and $(1+a)N = 0$. But A is an integral domain, thus $N = 0$. □

Geometrical interpretation;

The only function which vanishes to arbitrarily high order at a point is the zero function.

# October, 8, 2020

We study composition series leading up to Jordan-Hölder theory. Then we look at Artinian rings and Akizuki's Theorem.

9.4-9.5

Definition

*A finite ascending chain* $\{M_i\}$ *in an A-module M, which begins at the zero module* 0 *and ends at M, is called a* **composition series** *if all its subquotients* $M_{i+1}/M_i$ *are simple modules.*

(By convention simple modules are non-zero, so in particular all the inclusions $M_i \subset M_{i+1}$ are strict. The series when displayed appears like

$$0 = M_0 \subset M_1 \subset \ldots \subset M_{n-1} \subset M_n = M,$$

where each subquotient $M_{i+1}/M_i$ is shaped like $A/\mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i$ in $A$.)

Definition

*The number $n$ is called the* **the length** *of the series; it is the number of non-zero constituencies. More generally,* **the length** *of any finite chain will be the number of inclusions; that is, one less than the number of modules.*

For a finite chain being a composition series is equivalent to being a maximal chain; that is, no module can be inserted to make it longer.

* Contains 0 and $M$
* No submodules lying strictly between two consecutive terms.
* Let $\beta\colon M \to M'$ be an $A$-linear map and $\mathscr{M} = \{M_i\}$ a composition series in $M$. The set $\{\beta(M_i)\}$ of images is obviously a chain in $M'$, but inclusions do not necessarily persist being strict, so there may be repetitions in $\{\beta(M_i)\}$. A part from that, $\{\beta(M_i)\}$ will be a composition series:

### Theorem (Weak Jordan-Hölder)

*Assume that $M$ has a composition series. Then all composition series in $M$ have the same length and any chain may be completed to a composition series.*

The common length of the composition series is called the **length** of the module and denoted $\operatorname{length}_A M$. For modules not of finite length, that is those having no composition series, one naturality writes $\operatorname{length}_A M = \infty$. As a matter of pedantry, the zero module is considered to be of finite length and its length is zero (what else?).

A closer look at the proof above reveals that it in fact gives the full Jordan-Hölder theorem:

### Theorem (True Jordan-Hölder)

*Any two composition series of a module of finite length have up to order the same subquotients.*

### Proposition (Additivity of length)

*Given a short exact sequence of A-modules*

$$0 \to M' \longrightarrow M \longrightarrow M'' \to 0$$

*Then $M$ is of finite length if and only if the two others are, and it holds true that $length_A M = length_A M' + length_A M''$.*

An immediate corollary of is that modules of finite length are both Noetherian and Artinian. Obviously this is true for simple modules (no submodules, no chains) and hence follows in general by a straightforward induction on the length using Proposition . The converse holds as well:

### Proposition

*An A-module M is of finite length if and only if it is both Noetherian and Artinian.*

*Proof.* Assume $M$ both Noetherian and Artinian. Since $M$ is Artinian every non-empty set of submodules has a minimal element, so if $M$ is not of finite length, there is a submodule, $N$ say, minimal subjected to being non-zero and not of finite length. It is finitely generated because $M$ is Noetherian and hence Nakayama's lemma applies: There is surjection $\phi\colon M \to k$ onto a simple module $k$. The kernel of $\phi$ is of finite length by the minimality of $N$, and hence $N$ itself is of finite length by proposition above. □

Be aware that the base ring $A$ is a serious part of the game and can have a decisive effect on the length of a module. If $A \to B$ is a map of rings and $M$ a $B$-module which is of finite length over both $A$ and $B$, there is in general no reason that $\mathrm{length}_A M$ and $\mathrm{length}_B M$ should agree. Already when $k \subset K$ is a finite non-trivial extension of fields the two lengths differ in that $\dim_K V = [K;k] \cdot \dim_k V$ for vector spaces over $K$.

Unlike what is true for vector spaces, module of the same length need not be isomorphic. A simple example are the $\mathbb{Z}$-modules $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for different primes $p$. They are all of length one but not two are isomorphic!

## Example (Vector spaces)

*Over fields, of course the length of modules coincide with their vector space dimensions. In a similar fashion, if the $A$-module $M$ is killed by a maximal ideal $\mathfrak{m}$ in $A$, and therefore is a vector space over the field $A/\mathfrak{m}$, one has $length_A M = \dim_{A/\mathfrak{m}} M$.*

## Example

*Let $A = k[x,y]$ and $\mathfrak{m} = (x,y)$. For each non-negative integer $n$ let $M_n = k[x,y]/\mathfrak{m}^n$. Then there are short exact sequences*

$$0 \to \mathfrak{m}^{n-1}/\mathfrak{m}^n \longrightarrow M_n \longrightarrow M_{n-1} \to 0,$$

*so that $length_A M_n = length_A M_{n-1} + \dim_k \mathfrak{m}^{n-1}/\mathfrak{m}^n$. The module $\mathfrak{m}^{n-1}/\mathfrak{m}^n$ is a vector space over the field $A/\mathfrak{m} = k$ having the classes of the monomials $x^i y^{n-1-i}$ for $0 \leq i \leq n-1$ as a basis, and hence $length_A \mathfrak{m}^{n-1}/\mathfrak{m}^n = \dim_k \mathfrak{m}^{n-1}/\mathfrak{m}^n = n$. We conclude that $length_A M_n = length_A M_{n-1} + n$ and induction on $n$ yields that*

$$length_A M_n = \sum_{i=1}^{n} i = \binom{n+1}{2}.$$

## Proposition

*A finitely generated module M over a Noetherian ring is of finite length if and only if its support $\mathrm{Supp} M$ is a finite union of closed points.*

*Proof.* Assume to begin with that $M$ is of finite length and let $\{M_i\}$ be a composition series. Then citing Proposition we infer that $\mathrm{Supp} M = \bigcup_i \mathrm{Supp} M_i/M_{i+1} = \bigcup_i \{\mathfrak{m}_i\}$ where $M_i/M_{i+1} \simeq A/\mathfrak{m}_i$ are the subquotients of the composition series $\{M_i\}$. So the support is a finite union of closed points. For the other implication we resort to the Structure Theorem assuring there is a descending chain $\{M_i\}$ of submodules in $M$ whose subquotients are shaped like $A/\mathfrak{p}_i$ with $\mathfrak{p}_i$ being prime. Again by Proposition it holds that $\mathrm{Supp} M = \bigcup_i V(\mathfrak{p}_i)$. Now, if $\mathrm{Supp} M$ consists of finitely many closed points, all the prime ideals $\mathfrak{p}_i$'s will be maximal and consequently all the subquotients $M_i/M_{i-1}$ will be fields. Hence $M$ is of finite length (in fact, the chain $\{M_i\}$ will be a composition series). □

### Corollary (Structure of finite length modules)

*Assume that $M$ is a module of finite length over the ring $A$. Then there is a canonical isomorphism*

$$M \simeq \bigoplus_{\mathfrak{m} \in Supp M} M_{\mathfrak{m}}.$$

*Proof.* For each maximal ideal $\mathfrak{m}$ there is a localization maps $\iota_{\mathfrak{m}} \colon M \to M_{\mathfrak{m}}$ and we may combine them into a map $\phi \colon M \to \bigoplus_{\mathfrak{m} \in \mathrm{Supp} M} M_{\mathfrak{m}}$. □

The proof will be an application of the local to global principle, more precisely that being an isomorphism is a local property of linear maps . Our task is therefore to establish that the localizations $\phi_{\mathfrak{m}}$ of $\phi$ are isomorphisms for all $\mathfrak{m}$. To cope with the double localizations that appear, we have the following lemma:

**Lemma**

*Let $M$ be a module of finite length over a rinf $A$ and let the maximal ideal $\mathfrak{m}$ belong to $Supp M$. Then $M_{\mathfrak{m}}$ is of finite length and has support $\{\mathfrak{m}\}$.*

*Proof.* [of the lemma] Let $\{M_i\}$ be a composition series in $M$ with $M_i/M_{i-1} \simeq A/\mathfrak{m}_\mathfrak{i}$. Now we contend that

$$(A/\mathfrak{m}_\mathfrak{i})_\mathfrak{m} = \begin{cases} 0 & \text{when } \mathfrak{m} \neq \mathfrak{m}_\mathfrak{i}; \\ A/\mathfrak{m} & \text{when } \mathfrak{m} = \mathfrak{m}_i. \end{cases}$$

Indeed, since $\mathfrak{m} \neq \mathfrak{m}_i$ is equivalent to $\mathfrak{m}_i \not\subseteq \mathfrak{m}$ (both are maximal ideals), i.e. to there being elements in $\mathfrak{m}_i$ not in $\mathfrak{m}$ (so some element killing $A/\mathfrak{m}_i$ gets inverted in the localization), it holds that $(A/\mathfrak{m}_i)_\mathfrak{m} = 0$ when $\mathfrak{m} \neq \mathfrak{m}_i$. Furthermore, it obviously holds that $(A/\mathfrak{m})_\mathfrak{m} = A/\mathfrak{m}$ (elements not in $\mathfrak{m}$ act invertibly on the field $A/\mathfrak{m}$). We infer that after possible repetitions are discarded, the chain $\{(M_i)_\mathfrak{m}\}$ is a composition series in $M_\mathfrak{m}$ with all subquotients equal to $A/\mathfrak{m}$. □

With this lemma up our sleeve, it follows painlessly that $\phi_{\mathfrak{m}}$ is an isomorphism for all $\mathfrak{m}$. To fix the ideas let $\mathrm{Supp} M = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$. If $\mathfrak{m} \notin \mathrm{Supp} M$, we have $M_{\mathfrak{m}} = 0$ and $(\bigoplus M_{\mathfrak{m}_i})_{\mathfrak{m}} = 0$, so $\phi_{\mathfrak{m}}$ is the zero map (which is an isomorphism in this case). If $\mathfrak{m}$ is one of the $\mathfrak{m}_i$'s, say $\mathfrak{m} = \mathfrak{m}_j$, the lemma gives

$$\Big(\bigoplus_i M_{\mathfrak{m}_\mathrm{i}}\Big)_{\mathfrak{m}} = \Big(\bigoplus_i M_{\mathfrak{m}_\mathrm{i}}\Big)_{\mathfrak{m}_j} \simeq \bigoplus_i \big(M_{\mathfrak{m}_i}\big)_{\mathfrak{m}_j} = M_{\mathfrak{m}_\mathrm{j}},$$

and $\phi_{\mathfrak{m}} : M_{\mathfrak{m}} \to M_{\mathfrak{m}}$ is the identity map.

**Theorem (Akizuki's Theorem)**

*An Artinian ring is Noetherian. Hence $A$ has only finitely many prime ideals which all are maximal.*

The proof of Akizuki's theorem is organized as a sequence of three lemmas. The first is about Artinian domains:

**Lemma**

*An Artinian domain $A$ is a field.*

*Proof.* Let $f \in A$ be a non-zero member of $A$. The principal ideals $(f^i)$ form a descending chain which must be ultimately constant; that is, $(f^{\nu+1}) = (f^{\nu})$ for some $\nu$. Then $f^{\nu} = af^{\nu+1}$ for some $a \in A$, and cancelling $f^{\nu}$ which is permissible as $A$ is a domain, we find $1 = af$; i.e. $f$ is invertible. □

## Lemma

*An Artinian ring $A$ has only finitely many prime ideals, and they are all maximal. Hence, if $J$ denotes the radical of $A$, the quotient $A/J$ is a finite product of fields.*

*Proof.* We have already established the first assertion, if $\mathfrak{p}$ is a prime in $A$, the quotient $A/\mathfrak{p}$ is an Artinian domain, hence a field by Lemma above. As to the second statement, assume that $\{\mathfrak{m}_i\}_{i\in\mathbb{N}}$ is a countable set of different maximal ideals in $A$. For each natural number $r$ consider the ideal $N_r = \bigcap_{i\leq r}\mathfrak{m}_i$. They form a descending chain, and $A$ being Artinian it holds true that $N_n = N_{n+1}$ for some $n$. This means that $\bigcap_{i\leq r}\mathfrak{m}_i \subset \mathfrak{m}_{n+1}$, and by Proposition , one of the $\mathfrak{m}_i$'s must lie within $\mathfrak{m}_{n+1}$, contradicting the assumption that the $\mathfrak{m}_i$'s are different.

The last assertion ensues from the Chinese remainder theorem. If $\mathfrak{m}_1,\ldots,\mathfrak{m}_r$ are the prime ideals in $A$, the radical $J$ equals $J = \bigcap_i \mathfrak{m}_i$, and since all the $\mathfrak{m}_i$'s are maximal, they are pair-wise comaximal. Hence the Chinese Remainder Theorem gives an isomorphism $A/J \simeq \prod_i A/\mathfrak{m}_i$. □

### Lemma

*The radical $J$ of an Artinian ring $A$ is nilpotent; that is, $J^n = 0$ for some $n$. Moreover, $J$ is Noetherian.*

This lemma concludes the proof of Akizuki's theorem. Since $A/J$, being the product of finite number of fields is Noetherian, we infer that $A$ is Noetherian. *Proof.* The descending chain of powers $\{J^\nu\}$ becomes stationary at a certain stage; that is, there is an $r$ such that $J^{r+1} = J^r$. Putting $\mathfrak{a} = \mathrm{Ann} J^r$ one finds

$$(\mathfrak{a} : J) = \{x \,|\, xJ \subset \mathrm{Ann} J^r\} = \mathrm{Ann} J^{r+1} = \mathfrak{a}.$$

If $\mathfrak{a} = A$, then $J^r = 0$ and we are happy, so assume that $\mathfrak{a}$ is a proper ideal, and let $\mathfrak{b}$ be a minimal ideal strictly containing $\mathfrak{a}$; such exist since $A$ is Artinian. Let $x \in \mathfrak{b}$ but $x \notin \mathfrak{a}$, then $\mathfrak{b} = \mathfrak{a} + Ax$. If $\mathfrak{a} + Jx =$, it follows that $Ax/Ax \cap \mathfrak{a} = I \cdot Ax/Ax \cap \mathfrak{a}$, and Nakayama's lemma yields that $Ax \subset \mathfrak{a}$ which is not the case. Hence $\mathfrak{a} + Jx$ is strictly contain in $\mathfrak{b}$, and by minimality $\mathfrak{a} + Jx = \mathfrak{a}$. Hence $x \in (\mathfrak{a} : J) = \mathfrak{a}$, whichis a contradiction. $\square$

The final step of the proof of Akizuki's theorem is an induction argument to show that $J$ is Noetherian. For $\nu$ sufficiently big, we saw above that $J^\nu = 0$, and for each $\nu$ there is a short sequence:

$$0 \to J^{\nu+1} \longrightarrow J^\nu \longrightarrow J^\nu / J^{\nu+1} \to 0$$

Submodules and quotients of Artinian modules are Artinian so it follows that $J^\nu$, and therefore also $J^\nu / j^{\nu+1}$, is Artinian. But $J^\nu / J^{\nu+1}$ is a module over $A/J$ which we just proved is a finite product of fields, and over such rings any Artinian module is Noetherian and we are through by descending induction on $\nu$.

**Theorem**

*Let $A$ be an Artinian ring. Then $\mathit{Spec}A$ is finite and discrete, and the localisation maps $A \to A_{\mathfrak{m}}$ induce an isomorphism*

$$A \simeq \prod_{\mathfrak{m} \in \mathit{Spec}A} A_{\mathfrak{m}}.$$

*If $A$ is Noetherian and $\mathit{Spec}A$ is finite and discrete, then $A$ is Artinian.*

Saying $\mathrm{Spec}A$ is finite and discrete is just another way of saying that all prime ideals in $A$ are maximal and finite in number. Anticipating the notion of **Krull dimension**, a ring all whose prime ideals are maximal is said to be of Krull dimension zero. Hence a Noetherien ring $A$ is Artinian if and only if its Krull dimension equals zero.