# MAT 4200 Commutative algebra

## Part 4, Primary decomposition (Chapt. 10)

October, 14, - October, 15, 2020

# October, 14, 2020

In this section we state and prove the Lasker-Noether Theorem.

10.1-10.2

**Definition**

Alt. 1: *An ideal $\mathfrak{q} \subset A$ is said to be* **primary** *if for every element $x \in A$ the multiplication map $A/\mathfrak{q} \to A/\mathfrak{q}$ that sends an element $y$ to $x \cdot y$ is either injective or nilpotent.*

Alt. 2: *An ideal $\mathfrak{q} \subset A$ is said to be* **primary** *if for any $x, y \in A$ such that $xy \in \mathfrak{q}$, then either $y \in \mathfrak{q}$ or $x \in \sqrt{\mathfrak{q}}$.*

### Example

*If $f$ is an irreducible element in the unique factorization domain $A$, then principal ideals $(f^n)$ generated by a power of $f$ are primary.*

### Proposition

*If $\mathfrak{q}$ is a primary ideal in the ring $A$, the radical $\sqrt{\mathfrak{q}}$ is a prime ideal, and it is the smallest prime ideal containing $\mathfrak{q}$.*

*Proof.* Assume that $xy \in \sqrt{\mathfrak{q}}$, but $y \notin \sqrt{\mathfrak{q}}$; then $x^n y^n$ lies in $\mathfrak{q}$ for some $n$, but $y^n \notin \mathfrak{q}$, so some power of $x^n$ lies there. Hence $x \in \sqrt{\mathfrak{q}}$. □

It is customary to say that a primary ideal $\mathfrak{q}$ is $\mathfrak{p}$-**primary** when $\mathfrak{p} = \sqrt{\mathfrak{q}}$, which also is phrased as $\mathfrak{p}$ **belongs to** $\mathfrak{q}$. The converse of the above proposition does not hold in general; the radical being prime is not sufficient for an ideal to be primary.

## Proposition

*An ideal $\mathfrak{q}$ whose radical is maximal, is primary.*

*Proof.* Assume that the radical $\sqrt{\mathfrak{q}}$ is maximal and write $\mathfrak{m} = \sqrt{\mathfrak{q}}$. Because $\mathfrak{m}$ is both maximal and the smallest prime containing $\mathfrak{q}$, the ring $A/\mathfrak{q}$ is a local ring with maximal ideal $\mathfrak{m}/\mathfrak{q}$ as the only prime ideal. Therefore the elements of $\mathfrak{m}/\mathfrak{q}$ are nilpotent while those not in $\mathfrak{m}/\mathfrak{q}$ are invertible. □

## Corollary

*The powers $\mathfrak{m}^n$ of a maximal ideal $\mathfrak{m}$ are $\mathfrak{m}$-primary.*

*Proof.* The radical of $\mathfrak{m}^n$ equals $\mathfrak{m}$. □

**Lemma**

*Assume that $B \subset A$ is an extension of rings and that $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal in $A$. Then $\mathfrak{q} \cap B$ is a $\mathfrak{p} \cap B$-primary ideal in $B$.*

*Proof.* Since $B/\mathfrak{q} \cap B \subset A/\mathfrak{q}$, the multiplication by $x$ in $B/\mathfrak{q} \cap B$ is either injective or nilpotent since this holds for multiplication by $x$ in $A/\mathfrak{q}A$. That $(\sqrt{\mathfrak{q}}) \cap B = \sqrt{(\mathfrak{q} \cap B)}$ is trivial. □

## Example

*The ideal $\mathfrak{a} = (x^2, xy)$ in the polynomial ring $k[x, y]$ has a radical that is prime, but $\mathfrak{a}$ is not primary. The radical of $(x^2, xy)$ equals $(x)$, which is prime, but in the quotient $k[x, y]/(x^2, xy)$ multiplication by $y$ is neither injective nor nilpotent ($y$ kills the class of $x$, but no power of $y$ lies in $(x^2, xy)$). One decomposition of $(x^2, xy)$ as an intersection of primary ideals is*

$$(x^2, xy) = (x) \cap (x^2, y).$$

*Checking the equality is not hard. One inclusion is trivial, and the other holds since a relation $z = ax = bx^2 + cy$ implies that $x|c$ (the polynomial ring is UFD, and hence $z \in (x^2, xy)$. Notice that both ideals in the intersections are primary; $(x)$ since it is prime and $(x^2, y)$ because the radical equals $(x, y)$ which is maximal.*

### Example

*One also has other decompositions of $\mathfrak{a}$ into an intersection of primary ideals, for instance, it holds true that*

$$(x^2, xy) = (x) \cap (x, y)^2.$$

*Indeed, the polynomials in $(x^2, xy)$ are those with $x$ as factor that vanish at least to the second order at the origin. This exemplifies that primary decompositions are not unique in general.*

## Example

*This is the standard example of a prime ideal whose square (or any its powers, for that matter) is not primary. It goes as follows: Let* $A = k[X,Y,Z]/(Z^2 - XY)$ *and, true to our usual convention we denote byx,* $y$ *and* $z$ *the classes of the variables in* $A$*; then the constituting relation* $z^2 = xy$ *holds in* $A$*. The ideal* $\mathfrak{p} = (z,x)$ *is prime, but* $\mathfrak{p}^2$ *is not primary; indeed,* $yx$ *lies there, but neither does* $x$ *lie in* $\mathfrak{p}^2$ *nor does* $y$ *lie in* $\mathfrak{p}$*. A decomposition of* $\mathfrak{p}^2$ *into primary ideals is shaped like*

$$(z,x)^2 = (z^2, zx, x^2) = (yx, zx, x^2) = (x,y,z) \cap (x).$$

*Obviously the ideal* $(x,y,z)$ *being maximal is primary. The ideal* $(x)$ *is more interesting. Killing* $x$*, we obtain the ring* $A/(x) = k[Y,Z]/(Z^2)$*, whose elements are either non-zero divisors or nilpotent and* $(x)$ *is a primary ideal. It's radical equals* $(z,x)$*.*

### Proposition

*If $q_i$ is a finite collection of $\mathfrak{p}$-primary ideals, then the intersection $\bigcap_i \mathfrak{q}_i$ is $\mathfrak{p}$-primary.*

*Proof.* Taking radicals commutes with taking finite intersection and therefore one has $\sqrt{\bigcap_i \mathfrak{q}_i} = \bigcap_i \sqrt{\mathfrak{q}_i} = \mathfrak{p}$. Assume next that $xy \in \bigcap_i \mathfrak{q}_i$, but $y \notin \bigcap_i \mathfrak{q}_i$; that is, $xy \in \mathfrak{q}_i$ for each $i$, but $y \notin \mathfrak{q}_\nu$ for some $\nu$. Since $\mathfrak{q}_\nu$ is $\mathfrak{p}$-primary $x$ lies in the radical $\sqrt{\mathfrak{q}}_\nu$ of $\mathfrak{q}_\nu$, which equals $\mathfrak{p}$, but as we just checked, $\mathfrak{p}$ is as well the radical of the intersection $\bigcap_i \mathfrak{q}_i$. □

The hypothesis that the intersection be finite cannot be discarded. Powers $\mathfrak{m}^i$ of a maximal ideal are all primary and have the same radical, namley $\mathfrak{m}$, but at least when $A$ is a Noetherian domain, their intersection equals the zero ideal (0) by Krull's intersection theorem — which might be primary, but certainly not $\mathfrak{m}$-primary (in most cases). There are however instances when infinite intersection of primary ideals are primary.

## Proposition

*Let $S$ a multiplicative set in the ring $A$ and let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal. Assume that $S \cap \mathfrak{p} = \emptyset$. Then $S^{-1}\mathfrak{q}$ is $S^{-1}\mathfrak{p}$-primary and it holds true that $\iota_S^{-1}(S^{-1}\mathfrak{q}) = \mathfrak{q}$.*

*Proof.* Localizing commutes with forming radicals so the radical of $S^{-1}\mathfrak{q}$ equals $S^{-1}\mathfrak{p}$. Assume that $x/s \cdot y/s' \in S^{-1}\mathfrak{q}$, but that $y/s' \notin S^{-1}\mathfrak{q}$. Then $txy \in \mathfrak{q}$ for some $t \in S$, and obviously it holds that $y \notin \mathfrak{q}$. Hence $tx$ lies in the radical $\mathfrak{p}$ of $\mathfrak{q}$, and since $t \notin \mathfrak{p}$, we conclude that $x \in \mathfrak{p}$; in other words $x/s$ lies in $S^{-1}\mathfrak{p}$.
To verify that $\iota_S^{-1}(S^{-1}\mathfrak{q}) = \mathfrak{q}$ let $x \in A$ be such that $\iota_S(x) \in S^{-1}\mathfrak{q}$. This means that $sx \in \mathfrak{q}$ for some $s \in S$, but by hypothesis $\mathfrak{p} \cap S = \emptyset$ so that $s \notin \mathfrak{p}$; thence $x \in \mathfrak{q}$ because $\mathfrak{q}$ is primary. □

## Proposition

*Let $\phi : A \to B$ be a surjective map of rings with kernel $\mathfrak{a}$. Assume that $\mathfrak{q}$ is an ideal in $A$ containing the kernel $\mathfrak{a}$. Then the image $\mathfrak{q}B = \mathfrak{q}/\mathfrak{a}$ of $\mathfrak{q}$ in $B$ is primary if and only if $\mathfrak{q}$ is. The radical of the image equals the image of the radical; or in symbols, $\sqrt{(}\mathfrak{q}/\mathfrak{a}) = (\sqrt{\mathfrak{q}})/\mathfrak{a}$.*

*Proof.* This is pretty obvious because by the Isomorphism Theorem it holds that $A/\mathfrak{q} = B/\mathfrak{q}B$, so the multiplication-by-what-ever-maps are the same. □

In particular, we observe that the ideal $\mathfrak{q}$ is primary if and only if the zero ideal $(0)$ is a primary ideal in the quotient $A/\mathfrak{q}$.

## Definition

1. *Let $\mathfrak{a}$ be an ideal in the ring $A$. A* **primary decomposition** *of $\mathfrak{a}$ is an expression of $\mathfrak{a}$ as a finite intersection of primary ideals; that is, an equality like*

$$\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$$

*where the $\mathfrak{q}_i$'s are primary ideals.*

2. *The primary decomposition is called* **minimal** *or* **reduced** *if all the radicals $\sqrt{\mathfrak{q}_i}$ are different and the intersection is irredundant.*

### Example

*Consider* $\mathfrak{m}^2 = (x^2, xy, y^2)$ *in the polynomial ring* $k[x,y]$ *(where* $\mathfrak{m} = (x,y)$*). For all scalars* $\alpha$ *and* $\beta$ *with* $\alpha \neq 0$ *one has the equality*

$$\mathfrak{m}^2 = (x^2, xy, y^2) = (x^2, y) \cap (y^2, \alpha x + \beta y).$$

*Indeed, one easily checks that* $\mathfrak{m}^2 \subset (x^2, y) \cap (y^2, \alpha x + \beta y)$ *(since* $\alpha \neq 0$*), and the other inclusion amounts to the two lines generated by the class of* $y$ *and the class of* $\alpha x + \beta y$ *in the two dimensional vector space* $\mathfrak{m}/\mathfrak{m}^2$ *intersecting in* $0$*.*

### Proposition

*Assume that $S$ is a multiplicatively closed subset of the ring $A$ and that $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$ is a primary decomposition of an ideal $\mathfrak{a}$ and denote the radical of $\mathfrak{q}_i$ by $\mathfrak{p}_i$. Then it holds true that $S^{-1}\mathfrak{a} = S^{-1}\mathfrak{q}_1 \cap \cdots \cap S^{-1}\mathfrak{q}_r$. Moreover, either $S^{-1}\mathfrak{q}_i$ is primary with radical $S^{-1}\mathfrak{p}_i$ or $S^{-1}\mathfrak{q}_i = S^{-1}A$.*

*Proof.*

1. Localization commutes with finite intersection.
2. If $S \cap \mathfrak{q} \neq \emptyset$, then $S^{-1}\mathfrak{q} = S^{-1}A$.
3. Assume $S \cap \mathfrak{q} = \emptyset$. Then we have shown that $S^{-1}\mathfrak{q}$ is $S^{-1}\mathfrak{p}$-primary.

□

The resulting decomposition of $S^{-1}\mathfrak{a}$ is not always irredundant even if the one one starts with is. The primes $\mathfrak{p}_i$ meeting $S$ blow up to the entire ring $S^{-1}A$ and will not contribute to the intersection; they can thus be discarded, and one may write

$$S^{-1}\mathfrak{a} = \bigcap_{S\cap\mathfrak{p}_i=\emptyset} S^{-1}\mathfrak{q}_i.$$

A particularly interesting case arises when one takes $S$ to be the complement of one of the $\mathfrak{p}_i$'s, say $\mathfrak{p}_\nu$. Then for $A$ localised at the prime ideal $\mathfrak{p}$, that is in the multiplicative set $A\setminus\mathfrak{p}$. $\mathfrak{a}A_{\mathfrak{p}_\nu} = \mathfrak{q}_\nu A_{\mathfrak{p}_\nu}$, and $\mathfrak{a}A_{\mathfrak{p}_\nu}$ is a *primary* ideal in $A_{\mathfrak{p}_\nu}$.

**Proposition**

*In a Noetherian ring A any ideal* $\mathfrak{a}$ *is the intersection of finitely many primary ideals.*

*Proof.*

1. Since the ring $A$ is assumed to be Noetherian, the set of ideals for which the conclusion fails, if non-empty, has a maximal element $\mathfrak{a}$. Replacing $A$ by $A/\mathfrak{a}$ we may assume that the zero ideal is not the intersection of finitely many primary ideals in particular it is not primary), but that all non-zero ideals in $A$ are, and aim for a contradiction.
2. By the new assumption in 1. , there will be two elements $x$ and $y$ in $A$ such that $xy = 0$, but with $x \neq 0$ and $y$ not nilpotent.
3. The different annihilators $\mathrm{Ann}(y^i)$ form an ascending chain of ideals, and hence $\mathrm{Ann}(y^\nu) = \mathrm{Ann}(y^{\nu+1})$ for some $\nu$.

□

*Proof.* [continue]

4. We contend that $(0) = \mathrm{Ann}(y) \cap (y^\nu)$. Indeed, if $a = by^\nu$ lies in $\mathrm{Ann}(y)$, one has $ay = by^{\nu+1} = 0$, therefore $b \in \mathrm{Ann}(y^{\nu+1}) = \mathrm{Ann}(y^\nu)$, and it follows that $a = by^\nu = 0$.
5. Now, $x \in \mathrm{Ann}(y)$ is a non-zero element, and since $y$ is not nilpotent, both ideals $(y^\nu)$ and $\mathrm{Ann}(y)$ are non-zero and are therefore finite intersections of primary ideals; The same obviously then holds true for $(0)$. Thus the zero ideal $(0)$ is not crooked, contradicting the assumption it were!

□

**Theorem (The Lasker-Noether theorem)**

*Every ideal in a Noetherian ring has a minimal primary decomposition.*

*Proof.* Start with any decomposition of an ideal $\mathfrak{a}$ into primary ideals (there is at least one according to the proposition above). By the above lemma it can be made minimal by regrouping ideals with the same radical and discarding redundant. □

# October, 15, 2020

We continue to study primary decompositions, in particular the uniqueness theorems.

10.3-10.5

## Proposition

*Let $\mathfrak{a}$ be an ideal in a Noetherian ring $A$. The radicals that occur in an irredundant primary decomposition of $\mathfrak{a}$, are precisely those transporter ideals $(\mathfrak{a} : x)$ with $x$ in $A$ that are prime.*

Passing to the quotient $A/\mathfrak{a}$ and observing that the quotient $(\mathfrak{a} : x)/\mathfrak{a}$ equals the annihilator $(0 : [x])$ of the class $[x]$ in $A/\mathfrak{a}$, the theorem has the equivalent formulation:

## Proposition (Principle of annihilators)

*The radicals arising from an irredundant primary decomposition of the zero ideal in a Noetherian ring $A$ are precisely those ideals among the annihilators $Ann(x)$ of elements $x$ from the ring that are prime.*

*Proof.*

1. Fix an irredundant primary decomposition of the zero ideal (0). There are two implications to prove, that the radicals of the primary ideals are annihilators, and that the annihilators are found in the decomposition.

2. We begin with letting $\mathfrak{q}$ be one of the components and letting $\mathfrak{p} = \sqrt{\mathfrak{q}}$ denote the radical, we aim at exhibiting an element $x$ such that $\mathfrak{p} = \mathrm{Ann}(x)$. Denote by $\mathfrak{c}$ the intersection of the components in the decomposition other than $\mathfrak{q}$. Then $\mathfrak{c} \cap \mathfrak{q} = 0$, but $\mathfrak{c} \neq 0$ since the decomposition is irredundant.

3. Let $x \in \mathfrak{c}$ be a non-zero element such that $\mathrm{Ann}(x)$ is maximal among the annihilators of non-zero elements of $\mathfrak{c}$. We contend that $\mathrm{Ann}(x) = \mathfrak{p}$, and begin with showing the inclusion $\mathrm{Ann}(x) \subset \mathfrak{p}$. Because $x \neq 0$, it holds that $x \notin \mathfrak{q}$, and hence $xy = 0$ implies that $y \in \mathfrak{p}$ as $\mathfrak{q}$ is $\mathfrak{p}$-primary.

□

*Proof.*

4. In order to show the other inclusion pick an $y \in \mathfrak{p}$ and assume that $xy \neq 0$. Some power of $y$ lies in $\mathfrak{q}$ and therefore kills $x$. Hence there is a natural number $n$ so that $y^n x = 0$, but $y^{n-1}x \neq 0$. By the maximality of $\mathrm{Ann}(x)$ it holds true that $\mathrm{Ann}(x) = \mathrm{Ann}(y^{n-1}x)$, and consequently $y \in \mathrm{Ann}(x)$ which contradicts the assumption that $xy \neq 0$.
5. For the reverse implication, assume that $\mathrm{Ann}(x)$ is a prime ideal. Let $I$ be the set of indices such that $\mathfrak{q}_i$ does not contain $x$ when $i \in I$. Then $\bigcap_{i \in I} \mathfrak{q}_i \subset \mathrm{Ann}(x)$, because

$$x \cdot \bigcap_{i \in I} \mathfrak{q}_i \subset \bigcap_{i \notin I} \mathfrak{q}_i \cdot \bigcap_{i \in I} \mathfrak{q}_i \subset \bigcap_{i \notin I} \mathfrak{q}_i \cap \bigcap_{i \in I} \mathfrak{q}_i = (0).$$

Consequently it holds true that the product of appropriate powers of the corresponding radicals $\mathfrak{p}_i$ is contained in $\mathrm{Ann}(x)$. Since $\mathrm{Ann}(x)$ is supposed to be prime, it follows that $\mathfrak{p}_\nu \subset \mathrm{Ann}(x)$ for at least one $\nu \in I$. On the other hand, it holds true that $(0) = x \cdot \mathrm{Ann}(x) \subset \mathfrak{q}_\nu$ from which ensues that $\mathrm{Ann}(x) \subset \mathfrak{p}_\nu$ because $\mathfrak{q}_\nu$ is $\mathfrak{p}_\nu$-primary and $x \notin \mathfrak{p}_\nu$.

### Theorem (The First Uniqueness Theorem)

*The radicals occurring in an irredundant primary decomposition of an ideal in a Noetherian ring are unambiguously determined by the ideal.*

### Definition

*The radicals of the primary components are called the* **associated prime ideals** *of* $\mathfrak{a}$*, and the set they constitute is denoted by* $Ass(A/\mathfrak{a})$ *In particular,* $Ass(A)$ *will be the set of prime ideals associated to zero.*

There are no inclusion relations between the components of an irredundant primary decomposition, but that does not exclude inclusion relations between the associated primes.

An example:

$$(x^2, xy) = (x) \cap (x, y)^2$$

with the associated primes being $(x)$ and $(x, y)$ and where $(x) \subset (x, y)$.

### Definition

*Associated prime ideal which are minimal in* $\mathrm{Ass}(A)$ *are called* **isolated** *ideals or components. The other are called* **embedded***.*

In the example above, $(x)$ is an isolated prime whilst $(x, y)$ is embedded. The notion of components in this setting stems from algebraic geometry.

## Proposition

*In a Noetherian ring $A$ the sets $Spec(A)$ and $\mathrm{Ass}(A)$ have the same minimal elements; in other words, the minimal primes of $A$ are precisely the isolated associated primes. In particular, there are finitely many minimal primes.*

*Proof.* The radical $\sqrt{0}$ of $A$ equals the intersection of all minimal primes in $A$. On the other hand, we just expressed the radical $\sqrt{0}$ as the intersection of the prime ideals minimal in $\mathrm{Ass}(A)$. When the intersections of two families of prime ideals are equal and there are no inclusion relations between members of either family, the families coincide. Hence the sets $\mathrm{Spec}(A)$ and $\mathrm{Ass}(A)$ have the same minimal primes. □

## Proposition

*The set of zero-divisors in a Noetherian ring $A$ equals $\bigcup_{\mathfrak{p}\in Ass(A)} \mathfrak{p}$, the union of the associated primes.*

*Proof.* Let $\mathrm{Ann}(z)$ be maximal among the annihilators of non-zero elements in $A$. Then $\mathrm{Ann} z$ is prime and hence an associated prime of $A$. Indeed, if $xyz = 0$ and $xz \neq 0$, it follows from the maximality of $\mathrm{Ann} z$ that $\mathrm{Ann}(z) = \mathrm{Ann}(xz)$ because obviously $\mathrm{Ann}(z) \subseteq \mathrm{Ann}(xz)$. Hence $y \in \mathrm{Ann}(z)$, and as any annihilator ideal is contained in a maximal one, we are through. □

We have seen that

$$(x^2, xy) = (x) \cap (x^2, y) = (x) \cap (x, y)^2,$$

and both $(x^2, y)$ and $(x, y)^2$ are minimal primary components. Thus the minimal primary components are not unique.

**Theorem (The second uniqueness theorem)**

*The isolated components of an ideal $\mathfrak{a}$ in a Noetherian ring $A$ are unambiguously defined by $\mathfrak{a}$.*

UNIVERSITETET I OSLO

*Proof.*

1. We shall concentrate on one of the isolated associated prime ideals $\mathfrak{p}$ of $\mathfrak{a}$, but the main player will be a component $\mathfrak{q}$ that belongs to one of the minimal primary decompositions of $\mathfrak{a}$ and has radical $\mathfrak{p}$.
2. The salient point is, as already announced, the equality

$$\mathfrak{q} = \iota^{-1}(\mathfrak{a}A_{\mathfrak{p}}), \tag{1}$$

from which the theorem ensues as isolated prime ideals are invariants of $\mathfrak{a}$.

□

*Proof.* [Continue]

3. To establish the equality one writes the decomposition of $\mathfrak{a}$ as $\mathfrak{a} = \mathfrak{q} \cap \bigcap_i \mathfrak{q}_i$ where the intersection extends over the primary components different from $\mathfrak{q}$. Localizing at $\mathfrak{p}$ one finds

$$\mathfrak{a}A_{\mathfrak{p}} = \mathfrak{q}A_{\mathfrak{p}} \cap \bigcap_i \mathfrak{q}_i A_{\mathfrak{p}} = \mathfrak{q}A_{\mathfrak{q}}$$

since the $\mathfrak{q}_i$'s blow up when localized; that is, $\mathfrak{q}_i A_{\mathfrak{p}} = A_{\mathfrak{p}}$. Indeed, since $\mathfrak{p}$ is isolated, $\mathfrak{p}_i \not\subseteq \mathfrak{p}$ holds for all $i$. Taking inverse images on both sides and using the fact that $\iota_S^{-1}(S^{-1}\mathfrak{q}) = \mathfrak{q}$ we conclude that $\iota^{-1}(\mathfrak{a}A_{\mathfrak{p}}) = \mathfrak{q}$.

□

### Example

*For any natural number $n$ the equality*

$$(x^2, xy) = (x) \cap (x^2, xy, y^n)$$

*holds true in the polynomial ring $k[x,y]$, and this is an example of infinitely many different minimal primary decompositions of the same ideal. Indeed, that the left side is included in the right is obvious; to check the other, let $a$ belong to the right side. Then*

$$a = p \cdot x = q \cdot x^2 + r \cdot y^n + sxy,$$

*with $p$, $q$ $r$ and $s$ polynomials in $k[x,y]$. It follows that $x$ divides $r$, and hence that $a \in (x^2, xy)$.*

## Example

*We shall analyse the familiar case of the intersection of two quadratic curves; the unit circle centred at* $(0,1)$ *and a standard parabola. So let* $\mathfrak{a} = (x^2 + (y-1)^2 - 1, y - x^2)$ *in* $k[x,y]$*, where* $k$ *is any field of characteristic different from 2. A standard manipulation shows that the common zeros of the two polynomials are the points* $(1,1)$*,* $(-1,1)$ *and* $(0,0)$*, and the same manipulations give*

$$\mathfrak{a} = (x^2 + (y-1)^2 - 1, y - x^2) = (x^2(x^2-1), y - x^2).$$

*Any prime ideal* $\mathfrak{p}$ *containing* $\mathfrak{a}$ *must contain either* $x$*,* $x-1$ *or* $x+1$*. It contains* $y$ *if* $x$ *lies in it, and because* $y - x^2 = y - (x+1)(x-1) - 1$*, one has* $y - 1 \in \mathfrak{p}$ *in the two other cases. We conclude that the* $(x,y)$*,* $(x-1,y-1)$ *and* $(x+1,y-1)$ *are the only prime ideals containing* $\mathfrak{a}$*; and since they all three are maximal, the associated primes are found among them, and there can be no embedded component.*

## Example (Continue)

*To determine the primary components of* $\mathfrak{a}$*, we localize. In the local ring* $A = k[x,y]_{(x+1,y-1)}$*, where both* $x$ *and* $x-1$ *are invertible, we obtain the equality*

$$\mathfrak{a}A = (x^2-1, y-x^2) = (x+1, y-x^2) = (x+1, y-1).$$

*In similar fashion, in* $B = \mathbb{C}[x,y]_{(x-1,y-1)}$ *both* $x$ *and* $x+1$ *are invertible, and one has*

$$\mathfrak{a}B = (x^2(x^2-1), y-x^2) = (x-1, y-1).$$

*Finally, in* $C = k[x,y]_{(x,y)}$ *both* $x+1$ *and* $x-1$ *have inverses, and we see that*

$$\mathfrak{a}C = (x^2, y).$$

*Since there are no embedded components, we conclude that*

$$\mathfrak{a} = (x-1, y-1) \cap (x+1, y-1) \cap (x^2, y).$$

## Example

*Consider the intersection of the "saddle surface" $S$ given in $k^3$ by $z = xy$ and the union of the $xy$-plane and the $xz$-plane, which has the equation $yz = 0$. The plane $z = 0$ intersects $S$ in the union of the $x$-axis and the $y$-axis, and the plane $y = 0$ in the $x$-axis. The $x$-axis thus appears twice in the intersection, which algebraically is manifested by the occurrence of a non-prime primary component in the decomposition of the ideal $\mathfrak{a} = (z - xy, zy)$. Because $zy = (z - xy)y + xy^2$, it holds that $\mathfrak{a} = (xy^2, z - xy)$, and consequently one finds*

$$\mathfrak{a} = (xy^2, z - xy) = (x, z - xy) \cap (y^2, z - xy)$$

*since $x$ and $y^2$ are relatively prime to $z - xy$. Now, $(x, z - xy) = (x, z)$ is a prime ideal, and $(y^2, z - xy)$ is $(y, z)$-primary (for instance, since $k[x, y, z]/(y^2, z - xy) \simeq k[x, y]/(y^2)$), so we have found the primary decomposition of $\mathfrak{a}$.*

### Definition

*A prime ideal* $\mathfrak{p}$ *is* **associated** *to the module* $M$ *if it is the annihilator of an element* $x$ *in the* $M$*; that is* $\mathfrak{p} = (0 : x)$*, and in concordance with the notation for rings we denote the the set of associated primes of* $M$ *by* $Ass(M)$*.*

### Proposition

*Let* $A$ *be a ring and* $M$ *a Noetherian* $A$*-module. Then* $Ass(M)$ *is finite and non-empty. The minimal primes in* $Ass(M)$ *and* $Supp(M)$ *coincide.*

The radical $\sqrt{(0)} = \{x \in A \,|\, x \text{ acts nil-potently on } M\}$ is an ideal, and is a prime ideal if $M$ is primary. Indeed, if $(xy)^\nu$ kills $M$ and no power of $y$ does, there is a $z \in M$ so that $y^n z \neq 0$; and since $x^n(y^n z) = 0$, it follows that $x$ is nilpotent as $M$ is primary.

**Proposition**

*Every Noetherian module has a primary decomposition*

*Proof.* Let $N$ be the maximal rascal, and replace $M$ by $M/N$. That is $M$ is not primary. Hence $xz = 0$ but the homothety by $x$ is not nilpotent. Let $\ker[x^i] \subset \ker[x^{i+1}]$ is an ascending chain, hence stabilizes at certain point, say $\nu$. We contend that $(0) = \ker[x] \cap x^\nu M$. Indeed, if $xz = 0$ and $z = x^\nu w$ it follows that $w \in \ker[x]^{\nu+1}$, hence in $\ker[x]^\nu$. By consequence $z = x^{\nu+1} w = 0$. □

Given a submodule $N$ of the module $M$, one has the transporter ideal $(N : M)$ consisting of ring elements that multiply $M$ into $N$; that is

$$(N : M) = \{x \in A \,|\, xM \subset N\}$$

and the radical $\sqrt{(N : M)}$ is called the radical of $N$ relative to $M$. The elements are the ring elements such a power multiplies $M$ into $M$. If $N = 0$, they consist of the ring elements inducing a nilpotent homothety on $M$.

One extends the notion of primary ideals to modules in the following fashion:

i) The homothety by $x$ on $M/N$ is either injective or nilpotent.

Or in other words

ii) if $xz \in N$ then either $z \in N$ or $xN \in (N : M)$ for some $n$

### Proposition

*Let $N \subset M$ be a module and a submodule.*

i) *If $N$ is primary, the radical $(N : M)$ is a primary ideal and the consequently the radical $\mathfrak{p} = \sqrt{(N : M)}$ is a prime ideal. One says that $N$ is $\mathfrak{p}$-primary.*

ii) *Finite intersections of $\mathfrak{p}$-primary submodules are $\mathfrak{p}$-primary.*

iii) *If $N$ is $\mathfrak{p}$-primary and $S \cap \mathfrak{p} = \emptyset$, then $S^{-1}N$ is $S^{-1}\mathfrak{p}$-primary submodule of $S^{-1}M$.*

iv) *Assume that $L \subset N$. Then $N$ is $\mathfrak{p}$-primary, then $N/L$ is a $\mathfrak{p}$-primary submodule of $M/L$.*