MAT 4200 Commutative algebra

Part 6, Hilbert's Nullstellensatz (Chapt. 12)

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Hilbert's Nullstellensatz is the key result in the fruitful synthesis of algebra and geometry that algebraic geometry is. It comes in half a dozen different formulations with different weight on the geometric or the algebraic aspects of the result, some are called "weak" and are apparently weaker than those called "strong", but of course, in the end they will all be equivalent.

Theorem (Hilbert's Nullstellensatz, most famous version)

Let $k = \overline{k}$ be an algebraic closed field. Then all maximal ideals in $k[x_1, \ldots, x_n]$ are of the form $(x_1 - a_1, \ldots, x_n - a_n)$ where $(a_1, \ldots, a_n)$ are points in k^n.

It thus establishes a one-to-one-correspondence between points in k^n; that is, geometry, and maximal ideals in $k[x_1, \ldots, x_n]$; that is algebra.

Forr any ideal $\mathfrak{a}$ in $k[x_1, \ldots, x_n]$ the **zero locus** of $\mathfrak{a}$, or **the closed algebraic subset** defined by $\mathfrak{a}$, is the subset $Z(\mathfrak{a}) \subset k^n$ of points where all the polynomials from $\mathfrak{a}$ vanish, or expressed in formulae:

$$Z(\mathfrak{a}) = \{(a_1, \ldots, a_n) \in k^n \,|\, f(a_1, \ldots, a_n) = 0 \text{ for all } f \in \mathfrak{a}\}.$$

The subsets $Z(\mathfrak{a})$ of k^n and $V(\mathfrak{a})$ in $\mathrm{Spec}(A)$ are closely related, but be careful not to confuse them: a point $(a_1, \ldots, a_n) \in k$ belongs to $Z(\mathfrak{a})$ if and only if the maximal ideal $(x_1 - a_1, \ldots, x_r - a_r)$ lies in $V(\mathfrak{a})$, however $V(\mathfrak{a})$ is a considerable larger set with all the prime ideals containing $\mathfrak{a}$ as members.

Recall the converse of the Z-construction: for any subset $S \subset k^n$ the polynomials that vanish along S, form an ideal $I(S)$ in the polynomial ring, and the Nullstellensatz describes the relation between these to constructions. A simple but basic observation is that polynomials belonging to the radical $\sqrt{\mathfrak{a}}$ of $\mathfrak{a}$ all vanish along $Z(\mathfrak{a})$, and therefore one has $\sqrt{\mathfrak{a}} \subset I(Z(\mathfrak{a}))$. The Nullstellensatz tells us that this inclusion is an equality. This is also called the *Strong Nullstellensatz* since it easily is seen to imply the other versions.

Theorem (Strong Nullstellensatz)

Let k be an algebraically closed field and $\mathfrak{a}$ an ideal in $k[x_1, \ldots, x_n]$. Then one has $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

We are now ready to start the Tour de Nullstellensatz.

Theorem (General Nullstellensatz)

Let A be a finitely generated algebra over a field k and let $\mathfrak{m}$ be a maximal ideal in A. Then $A/\mathfrak{m}$ is a finite field extension of k.

Proof. The field $K = A/\mathfrak{m}$ is finitely generated as an k-algebra since A is. If it is not algebraic, it has transcendence degree at least one over k, say r, and by Noether's Normalization Lemma it is an integral extension of a polynomial ring $k[y_1, \ldots, y_r] \subset K$. Since K is a field, it follows that $k[y_1, \ldots, y_r]$ is a field, which is impossible since polynomial rings are not fields (if y is a variable, $1/y$ is certainly not a polynomial). Hence K is finite over the ground field k. □

Theorem (Weak Nullstellensatz II)

Let k be an algebraically closed field and let $\mathfrak{m}$ be a maximal ideal in the polynomial ring $k[x_1, \ldots, x_n]$. Then $\mathfrak{m}$ is of the form $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ for a point $a = (a_1, \ldots, a_n)$ in k^n.

Proof. By the general version of the Nullstellensatz above, the field $k[x_1, \ldots, x_r]/\mathfrak{m}$ is a finite extension of k, and is therefore equal to k since k is assumed to be algebraically closed. Let $\pi : k[x_1, \ldots, x_n] \to k$ be the ensuing quotient homomorphism. To retrieve the point a let $a_i = \pi(x_i)$. Then obviously all the polynomials $x_i - a_i$ lie in the kernel $\mathfrak{m}$ of π, and since a priori $(x_1 - a_1, \ldots, x_n - a_n)$ is a maximal ideal, it must be equal to $\mathfrak{m}$, and we are through. □

Theorem (Weak Nullstellensatz III)

Let k be algebraically closed and let $\mathfrak{a}$ be an ideal of the polynomial ring $k[x_1, \ldots, x_n]$. Then $Z(\mathfrak{a})$ is non-empty if and only if the ideal $\mathfrak{a}$ is a proper ideal.

Proof. Since $\mathfrak{a}$ is a proper ideal, there is a maximal ideal in $k[x_1, \ldots, x_n]$ containing $\mathfrak{a}$, which by the weak version II is of the form $(x_1 - a_1, \ldots, x_n - a_n)$. It follows that $(a_1, \ldots, a_n) \in Z(\mathfrak{a})$. □

Weak Nullstellensatz II $\Rightarrow$ Weak Nullstellensatz III

Proposition

Weak Nullstellensatz III $\Rightarrow$ *Weak Nullstellensatz II.*

Proof. If $\mathfrak{m}$ is a maximal deal in $k[x_1, \ldots, x_n]$, version III implies that $Z(\mathfrak{m})$ is non-empty, say $(a_1, \ldots, a_n) \in Z(\mathfrak{m})$. The maximal ideal $(x_1 - a_1, \ldots, x_n - a_n)$ will then contain $\mathfrak{m}$, but $\mathfrak{m}$ being maximal, the two must be equal. □

Weak Nullstellensatz II $\Leftrightarrow$ Weak Nullstellensatz III

We also have

Proposition

Strong Nullstellensatz $\Rightarrow$ Weak Nullstellensatz III.

Proof. Let $\mathfrak{a}$ be an ideal of the polynomial ring $R = k[x_1, \ldots, x_n]$. By the strong version we have $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. Now $Z(\mathfrak{a}) = \emptyset$ if and only if $I(Z(\mathfrak{a})) = R$, i.e. $\sqrt{\mathfrak{a}} = R$, but this is equivalent to $\mathfrak{a} = R$. □

Strong Nullstellensatz $\Rightarrow$ Weak Nullstellensatz III

The last stage in the Tour is a trick found by J.L Rabinowitsch and published in a thirteen lines long paper in 1929 which proves that the weak version III implies the full version.

Lemma

The weak version III implies that $I(Z(\mathfrak{a})) \subset \sqrt{\mathfrak{a}}$ *for all ideals in a polynomial ring* $k[x_1, \ldots, x_n]$.

Proof. The task is to demonstrate that $I(Z(\mathfrak{a})) \subset \sqrt{\mathfrak{a}}$ for any proper ideal $\mathfrak{a}$ in the polynomial ring $k[x_1, \ldots, x_n]$, under the assumption that the zero-locus $Z(\mathfrak{b})$ is non-empty whenever $\mathfrak{b}$ is a proper ideal in a polynomial ring.

1. The crux of the trick is to introduce an auxiliary variable x_{n+1} and for each element $g \in I(Z(\mathfrak{a}))$ to consider the ideal $\mathfrak{b}$ in the polynomial ring $k[x_1, \ldots, x_{n+1}]$, given by

$$\mathfrak{b} = \mathfrak{a} \cdot k[x_1, \ldots, x_{n+1}] + (1 - x_{n+1} \cdot g).$$

   In geometric terms, the zero-locus $Z(\mathfrak{b}) \subset \mathbb{A}^{n+1}(k)$ equals the intersection of the the subset $Z = Z((1 - x_{n+1} \cdot g))$ and the inverse image $\pi^{-1}\big(Z(\mathfrak{a})\big)$ of $Z(\mathfrak{a})$ under the projection $\pi : \mathbb{A}^{n+1}(k) \to \mathbb{A}^n(k)$ that forgets the auxiliary coordinate. This intersection is empty since obviously g does not vanish along Z, but vanishes identically on $\pi^{-1}\big(Z(\mathfrak{a})\big)$.

□

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Proof. [Continue]

2. According to the third third weak version the ideal $\mathfrak{b}$ is therefore not proper, so it holds that $1 \in \mathfrak{b}$, and there are polynomials f_i in $\mathfrak{a}$ and h_i and h in $k[x_1, \ldots, x_{n+1}]$ satisfying a relation like

$$1 = \sum f_i(x_1, \ldots, x_n) h_i(x_1, \ldots, x_{n+1}) + h \cdot (1 - x_{n+1} \cdot g).$$

3. Substituting $x_{n+1} = 1/g$ in this relation and multiplying through by a sufficiently high power g^N of g (for instance, the highest power of x_{n+1} that occurs in any of the h_i's will suffice) we obtain

$$g^N = \sum f(x_1, \ldots, x_n) H_i(x_1, \ldots, x_n),$$

where each $H_i(x_1, \ldots, x_n) = g^N \cdot h_i(x_1, \ldots, x_n, g^{-1})$ is an element in $k[x_1, \ldots, x_n]$. Hence $g \in \sqrt{\mathfrak{a}}$.

□

Corollary

The three versions of Hilbert Nullstellensatz, strong, weak II and weak III are all equivalent.

Strong Nullstellensatz $\Leftrightarrow$ Weak Nullstellensatz III $\Leftrightarrow$ Weak Nullstellensatz II

Proposition

Let A be an algebra finitely generated over the field k. Let $\mathfrak{a}$ be an ideal in A. Then the radical $\sqrt{\mathfrak{a}}$ equals the intersection of the maximal ideals containing $\mathfrak{a}$.

Proof. We prove the case that k is algebraically closed. The algebra A is by assumption the quotient of a polynomial ring $k[x_1, \ldots, x_n]$, and replacing $\mathfrak{a}$ by the inverse image in $k[x_1, \ldots, x_n]$, we may well assume that A is a polynomial ring. Now, a maximal ideal $(x_1 - a_1, \ldots, x_n - a_n)$ contains $\mathfrak{a}$ if and only if the point $(a_1, \ldots, a_n)$ belongs to $Z(\mathfrak{a})$. So if f lies in all these maximal ideals, it vanishes along $Z(\mathfrak{a})$, and by the Nullstellensatz, it lies in $\sqrt{\mathfrak{a}}$. □