UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT4200 — Commutative algebra

Day of examination: Friday, December 15, 2017

Examination hours: 14.30 – 18.30

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

EXERCISE 1 Let R be a ring, and M and N R-modules. Recall that the length $l(M)$ of a module is the infimum of the lengths of all composition series of submodules of M.

a) Explain what we mean by *additivity of length*.

b) Suppose $\phi : M \to N$ is an injective map of R-modules of finite length, and that $l(M) = l(N)$. Show that ϕ is an isomorphism.

EXERCISE 2 Let R be a ring, $S \subset R$ a multiplicatively closed set with $1 \in S$, $\mathfrak{a}$ an ideal. The *saturation* of the ideal $\mathfrak{a}$ with respect to S is denoted $\mathfrak{a}^S$ and defined by

$$\mathfrak{a}^S := \{a \in R \,|\, \exists s \in S \text{ such that } as \in \mathfrak{a}\}$$

a) Show that $\mathfrak{a}^S$ is an ideal, containing $\mathfrak{a}$.

b) Let $\mathfrak{p}$ be a prime ideal of R, and suppose $\mathfrak{p} \cap S = \emptyset$. Show that $\mathfrak{p}^S = \mathfrak{p}$.

c) Let $\phi_S : R \to S^{-1}R$ be the map sending $r \in R$ to $\frac{r}{1}$. Show that $\phi_S^{-1}(\mathfrak{a}S^{-1}R) = \mathfrak{a}^S$.

EXERCISE 3 Let k be a field, and R the graded ring given by

$$R = k[x, y, z]/(x^2 - y^2, y^2 - z^2)$$

a) Compute the Hilbert polynomial of R. What is the dimension of R?

b) Let $\mathfrak{m} = (x, y, z)$ be the maximal graded ideal of R, and (x) the principal ideal generated by x. Show that $\mathfrak{m}^3 \subseteq (x)$.

c) Show that (x) is a $\mathfrak{m}$-primary ideal in R with the least possible number of generators.

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Exercise 4 Let R be a Noetherian ring, $\mathfrak{p} \subset R$ a prime ideal and M a finitely generated R-module. The prime ideal $\mathfrak{p}$ is associated to M if there exists $x \in M$, $x \neq 0$ such that $Ann(x) = \mathfrak{p}$, and we write $\mathfrak{p} \in Ass(M)$. The set of prime ideals $\mathfrak{p} \subset R$ such that $M_{\mathfrak{p}} \neq 0$ is called the **support** of M, denoted $Supp(M)$.

a) Show that $Ass(M) \subseteq Supp(M)$.

Let R be a ring. An R-module M is said to be *simple* if for any submodule $N \subseteq M$ either $N = 0$ or $N = M$.

b) Show that if M is simple, then

$$M \cong R/\mathfrak{m}$$

for some maksimal ideal $\mathfrak{m} \subset R$.

c) Let M be a simple module. Show that $Supp(M)$ consists of only one element.

d) Recall that $\mathfrak{p} \in Supp(M)$ if and only if there is a prime ideal $\mathfrak{p}' \in Ass(M)$, such that $\mathfrak{p}' \subseteq \mathfrak{p}$.

Prove that if $dim(R) = 0$, then $Ass(M) = Supp(M)$.

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