UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT4200 — Commutative Algebra

Day of examination: Thursday, December 5th, 2019

Examination hours: 9:00 – 13:00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All rings are assumed to be commutative with 1.

Problem 1 (weight 15%)

a

Let A be a ring and let $f : M \to N$ be an A-linear map. Show that the following statements are equivalent:

(i) f is an isomorphism

(ii) $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is an isomorphism for every prime ideal $\mathfrak{p}$ in A

(iii) $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is an isomorphism for every maximal ideal $\mathfrak{m}$ in A

b

Suppose that A is an integral domain with fraction field K. Show that

$$A = \bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$$

where the intersection is taken over all maximal ideals of A. (Possible hint: Consider the map $A \to \bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$).

Problem 2 (weight 20%)

Consider the ring

$$A = \mathbb{C}[x, y, z]/(x^2 - y^2, z^2x - z^2y).$$

a

Find a minimal primary decomposition of the zero ideal in A.

(Continued on page 2.)

b

Compute the Krull dimension of A.

c

Describe all maximal ideals in A.

Problem 3 (weight 25%)

Let $A = \mathbb{C}[x, y, z, w]/I$, where I is the ideal generated by the 2×2-minors of the matrix

$$\begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}.$$

That is,

$$I = (xz - y^2, xw - yz, yw - z^2).$$

You may take for granted that I is a prime ideal. For simplicity, write x, y, z, w also for the images of the variables in A. Let $B = \mathbb{C}[x, w]$.

a

Show that A is free as a B-module:

$$A \simeq B \oplus By \oplus Bz.$$

b

Using the decomposition in a), or otherwise, compute the Hilbert polynomial of A.

c

Find a minimal primary decomposition of the ideal $J = I + (y)$, and find the associated prime ideals.

Problem 4 (weight 15%)

Let A be the integral closure of $\mathbb{Z}$ in $\mathbb{C}$.

a

Show that A is not noetherian (Hint: Consider the powers $2, 2^{1/2}, 2^{1/4}, 2^{1/8}, \ldots$).

b

Show that A has Krull dimension 1.

c

Is A a local ring?

(Continued on page 3.)

Problem 5 (weight 25%)

a

Let A be a local noetherian integral domain. Show that for a prime ideal $\mathfrak{p} \subset A$,

$$\dim(A_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) \leq \dim(A). \tag{1}$$

For the rest of the problem we will let $R = \mathbb{C}[t]_{(t)}$ and $A = R[x]$.
Let also $\mathfrak{p} = (xt - 1) \subset A$.

b

Compute the Krull dimension of A.

c

Show that $\mathfrak{p}$ is a maximal ideal.

d

Compute the height of $\mathfrak{p}$, $\mathrm{ht}(\mathfrak{p})$.
Conclude that the inequality in (1) is not an equality in general.

THE END