Let A be a non-zero ring and let Σ be the set of all multiplicative subsets S of A for which $0 \notin S$. Show that Σ has maximal elements and that $S \in \Sigma$ if and only if $A \setminus S$ is a minimal prime ideal of A.

Suppose that M and N are flat A-modules. Show that $M \otimes_A N$ is flat over A.

Let A be a Noetherian ring and $\phi : A \to A$ a ring homomorphism. Prove that if ϕ is surjective, then it is injective as well.

Let A be a domain, $\mathfrak{a}$ a nonzero ideal. Let K be the fraction field of A, i.e. $K = A_{(0)}$. Show that $\mathfrak{a} \otimes_A K = K$.

a) Let A be a ring, M a flat A-module, and B an A-algebra. Show that $M \otimes_A B$ is a flat B-algebra.
b) Let A be a ring, $\mathfrak{a}$ an ideal. Assume $A/\mathfrak{a}$ is A-flat. Show that $\mathfrak{a}^2 = \mathfrak{a}$.

Let A be an integral domain and $S = A \setminus \{0\}$. Let M be an A-module and consider the map $\iota_S : M \to S^{-1}M$, given by $x \mapsto \frac{x}{1}$. Let T be the torsion submodule of M

$$T(M) = \{x \in M \mid Ann(x) \neq 0\}$$

a) Show that $T(M) = ker(\iota_S)$.
b) Show that for any exact sequence $0 \to M' \to M \to M'' \to 0$ the sequence $0 \to T(M') \to T(M) \to T(M'') \to 0$ is exact too.

Let $\mathfrak{a}, \mathfrak{b}$ be ideals in a ring A. Suppose $\mathfrak{p}$ is a prime ideal such that $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$.

a) Show that
   1. $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$
   2. $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$
b) Show that the following three statements are equivalent:
   (i) $\mathfrak{a}\mathfrak{b} = \mathfrak{p}$
   (ii) $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{p}$
   (iii) $\mathfrak{a} = \mathfrak{p}$ or $\mathfrak{b} = \mathfrak{p}$

Let A be a ring.

a) Let $\mathfrak{m} \subset A$ be a maximal ideal. Prove that $\mathfrak{m}$ is a prime ideal.
b) Suppose that any element $x \in A$ satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Let A be a ring. Show that the following are equivalent:

i) A has exactly one prime ideal,
ii) every element of A is either a unit or nilpotent.

Let A be a ring and $\phi : A \to A$ a ring homomorphism.

a) Show that $B = \{x \in A \,|\, \phi(x) = x\}$ is a subring of A.
b) Show that if the composition $\phi^2(a) = \phi(\phi(a))$ is the identity map on A, then each element $a \in A$ is the root of a monic polynomial of degree two in $B[x]$.

Let A be a ring, S a multiplicatively closed subset of A and $\mathfrak{a}$ an ideal of A. The **saturation** of $\mathfrak{a}$ is the set

$$\mathfrak{a}^S = \{a \in A \,|\, \exists \in S \, \text{such that} \, as \in \mathfrak{a}\}$$

We call $\mathfrak{a}$ **saturated** if $\mathfrak{a}^S = \mathfrak{a}$.Prove the following statements:

i) $\ker(\iota : A \to S^{-1}A) = (0)^S$
ii) $\mathfrak{a} \subseteq \mathfrak{a}^S$
iii) $\mathfrak{a}^S$ is an ideal of A
iv) If $\mathfrak{a} \subseteq \mathfrak{b}$ are ideals of A, then $\mathfrak{a}^S \subseteq \mathfrak{b}^S$
v) $(\mathfrak{a}^S)^S = \mathfrak{a}^S$
vi) $(\mathfrak{a}^S\mathfrak{b}^S)^S = (\mathfrak{a}\mathfrak{b})^S$

Let A be a ring and $s \in A$ an element. Consider the multiplicatively closed subset $S = \{1, s, s^2, \dots\}$, and the corresponding map $\phi_S : A \to S^{-1}A$ given by $\phi_S(x) = \frac{x}{s}$.

a) Show that $S^{-1}A = 0$ if and only if s is nilpotent.
b) Show that ϕ_S is injective if and only if s is not a zero-divisor in A
c) Suppose A is local, with maximal ideal $\mathfrak{m}$, and that the Krull dimension of A is finite. Show that $S^{-1}A = A$ or that $\dim(S^{-1}A) < \dim(A)$.

a) Let R be a ring, S a multiplicatively closed subset. Prove that $S^{-1}R = 0$ if and only if S contains a nilpotent element.
b) Let R be a ring, S a multiplicatively closed subset, $\overline{S}$ its saturation, i.e.

$$\overline{S} = \{x \in R \,|\, \exists y \in R \, s.t. \, xy \in S\}$$

Let T be the set of non-zero elements of $S^{-1}R$. Show that

$$T = \{\frac{x}{s} \,|\, x \in \overline{S} \, and \, s \in S\}$$

c) Show that $\phi_S^{-1}(T) = \overline{S}$.
d) Let R be a ring, S a multiplicatively closed subset. Let $\mathbf{p}$ be a prime ideal in R, and assume $\mathbf{p} \cap S = \emptyset$. Then $\mathbf{p} = \mathbf{p}^S$ and $\mathbf{p}S^{-1}R$ is prime.

Let A be a ring and let M, N be two finitely generated A-modules such that $M \otimes_A N = 0$. Show that if A is a local ring, then either M or N is 0.

Let $A \subseteq B$ be two rings such that $B \setminus A$ is multiplicatively closed. Show that A is integrally closed in B.

a) La B være hel over A, begge integritetsområder, og $\mathbf{q}$ et ikke-trivielt primideal i B. Vis at $\mathbf{q} \cap A \neq 0$.
b) La B være hel over A, begge integritetsområder. Vi s at $dim(A) = dim(B)$.

a) Let k be a field. Define a ring homomorphism $f : k[x,y] \to k[t]$ by $x \mapsto t^2$, $y \mapsto t^5$. Show that this induces an injective homomorphism $A = k[x,y]/(x^5 - y^2) \to k[t]$.
c) Prove that the integral closure of A is isomorphic to $k[t]$.
b) Let $\mathbf{q} \subset k[t]$ be a non-zero prime ideal. Show that $f^{-1}(\mathbf{q})$ is a non-zero prime ideal. Prove that $dim(A) = 1$.

Gitt en gradert ring $A = k[x,y,z]/(x^2 - y^2, z^2 - x^2)$.

a) Regn ut Hilbertpolynomet til A. Hva er dimensjonen til A?
b) La $\mathbf{m} = (x,y,z)$. Finn et $\mathbf{m}$-primært ideal med færrest mulig generatorer.

Let k be a field, and A the graded ring given by

$$A = k[x,y,z]/(xy, xz, yz)$$

a) Compute the Hilbert-Poincaré series of A. What is the dimension of A?
b) Let $\mathbf{m} = (x,y,z)$ be the maximal graded ideal of A. Find an $\mathbf{m}$-primary ideal in A with the least possible number of generators.

Vi minner om definisjonen av et primært ideal: La $\mathbf{q} \subset A$ være et ideal. Vi sier at $\mathbf{q}$ er **primært** hvis og bare hvis følgende er oppfylt:
Dersom $xy \in \mathbf{q}$, så er enten $x \in \mathbf{q}$, eller $y \in r(\mathbf{q}) = \mathbf{p}$. Vi kan lage en annen definisjon, som vi for anledningen kan kalle Tony-primær: La $\mathbf{q} \subset A$ være et ideal. Vi sier at $\mathbf{q}$ er **Tony-primært** hvis og bare hvis følgende er oppfylt:
Dersom $xy \in \mathbf{q}$, så er enten $x \in r(\mathbf{q})$, eller $y \in r(\mathbf{q})$.

a) Vis at Primær medfører Tony-primær.
b) Forklar hvorfor eksemplet

$$\mathbf{q} = (\overline{x}, \overline{y})^2 \subset k[x,y,z]/(xz - y^2) = A$$

viser at den motsatte implikasjonen ikke gjelder, dvs. at idealet $\mathbf{q}$ er Tony-primært men ikke primært.

Let $R = k[x,y,z]$ be a polynomial ring in three variables over a field k. Let $\mathbf{p}_1 = (x,y)$, $\mathbf{p}_2 = (x,z)$ be two prime ideals and let $\mathbf{m} = (x,y,z)$ be a maximal ideal. Let $I = \mathbf{p}_1\mathbf{p}_2$. Show that $I = \mathbf{p}_1 \cap \mathbf{p}_2 \cap \mathbf{m}$ is a minimal primary decomposition of I. Which components are minimal and which are embedded?

a) Let $\mathbf{a} \subset R$ an ideal, and S a multiplicatively closed subset. Let $\phi : R \to S^{-1}(R/\mathbf{a})$ be the composition of $R \to R/\mathbf{a}$ and ϕ_S. Then $\mathbf{a}^S = \phi_S^{-1}(0)$. Show that $\mathbf{a} \subset \mathbf{a}^S = \mathbf{a}$ if and only if ...
b) Compute $\mathbf{a}^S$ for the composition

$$\phi : k[x,y] \to k[x,y)/(f) \to S^{-1}k[x,y)/(f)$$