

of ideals however, are not generally preserved, but the inclusions in the upcoming proposition are easily verified. One has:

PROPOSITION 2.11 (PULLBACKS) *Let $\phi: A \rightarrow B$ be a ring homomorphism and let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals in B . The pullback map $\phi^{-1}: \mathcal{I}(B) \rightarrow \mathcal{I}(A)$ preserves inclusions, and the following claims hold true:*

- i) $\phi^{-1}(\mathfrak{a}) \cap \phi^{-1}(\mathfrak{b}) = \phi^{-1}(\mathfrak{a} \cap \mathfrak{b})$;
- ii) $\phi^{-1}(\mathfrak{a}) + \phi^{-1}(\mathfrak{b}) \subseteq \phi^{-1}(\mathfrak{a} + \mathfrak{b})$;
- iii) $\phi^{-1}(\mathfrak{a}) \cdot \phi^{-1}(\mathfrak{b}) \subseteq \phi^{-1}(\mathfrak{a} \cdot \mathfrak{b})$.

Equality does not hold in general in the two last statements, but notice that the inclusion in ii) will be an equality when ϕ is a surjective map. For statement ii) we postpone giving an example to the end of the paragraph (Examples 2.7 below), but it is easy to find examples of strict inclusion in iii):

EXAMPLE 2.6 Most maps $\phi: A \rightarrow B$ with a non-trivial kernel will do: let both ideals $\mathfrak{a}$ and $\mathfrak{b}$ be the zero ideal in B . Then $\phi^{-1}(\mathfrak{a} \cdot \mathfrak{b}) = \ker \phi$, but of course, $\phi^{-1}(\mathfrak{a})^2 = (\ker \phi)^2$. This might be equal to $\ker \phi$, but in most cases is not, e.g. the kernel of the canonical map $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ has square (n^2) , which is different from (n) . Example 2.8 is an example that iii) fails even for injective maps. ★

(2.12) Next we come to the covariant construction. If $\mathfrak{a}$ is an ideal in A , the image $\phi(\mathfrak{a})$ is not necessarily an ideal in B unless ϕ is surjective. A stupid example can be the image of any non-zero ideal in $\mathbb{Z}$ under the inclusion $\mathbb{Z} \subseteq \mathbb{Q}$. The ideal generated by $\phi(\mathfrak{a})$ however is, and we shall usually denote this ideal by $\phi(\mathfrak{a})B$ or simply by $\mathfrak{a}B$; as mentioned above it is called the *extension* of $\mathfrak{a}$, but is also frequently referred to as the *pushout* of $\mathfrak{a}$. This induces a map $\mathcal{I}(A) \rightarrow \mathcal{I}(B)$. Inclusions are obviously preserved, and one leisurely verifies the other relations in the following proposition.

*Extension of an ideal
(utvidelse av et ideal)*

*Pushout of an ideal
(pushout, frempuff,
fremskudd)*

PROPOSITION 2.13 (PUSHOUTS) *Let $\phi: A \rightarrow B$ be a map of rings and let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals in A . Then the map $\mathfrak{a} \mapsto \mathfrak{a}B$ preserves inclusions. Moreover, the following hold true:*

- i) $\phi(\mathfrak{a} \cdot \mathfrak{b})B = \phi(\mathfrak{a})B \cdot \phi(\mathfrak{b})B$;
- ii) $\phi(\mathfrak{a} + \mathfrak{b})B = \phi(\mathfrak{a})B + \phi(\mathfrak{b})B$;
- iii) $\phi(\mathfrak{a} \cap \mathfrak{b})B \subseteq \phi(\mathfrak{a})B \cap \phi(\mathfrak{b})B$.

The inclusion in the last statement may be strict: just consider the map $\phi: k[x, y] \rightarrow k[z]$ between polynomial rings defined by the assignments $x \mapsto z$ and $y \mapsto z$. Then $\phi((x)) = \phi((y)) = (z)$, but $(x) \cap (y) = (xy)$ which maps to (z^2) . Example 2.10 on page 39 is a similar but more involved example.

Examples

(2.7) A simple example of strict inclusion in statement ii) in Proposition 2.11 above is the diagonal map $\delta: A \rightarrow A \times A$ that sends a to (a, a) . The two ideals $\mathfrak{b} = \{ (0, a) \mid a \in A \}$