Exactness of the tensor product and algebras

Let A be a ring, let M and N be A-modules. Recall we then have a module

$$M \otimes_A N,$$

the **tensor product** of M and N over A, which has the property that for an A-module P, we have

$$\mathrm{Hom}(M \otimes_A N, P) \leftrightarrow \{A - \text{bilinear maps } M \times N \to P\}.$$

Elements of $M \otimes_A N$ are of the form

$$\sum_{i=1}^{n} x_i \otimes y_i, \qquad x_i \in M, y_i \in N.$$

These satisfy relations, for all $x, x' \in M, y, y' \in N, a \in A$, that

$$(x + x') \otimes y = x \otimes y + x' \otimes y$$
$$x \otimes (y + y') = x \otimes y + x \otimes y'$$
$$ax \otimes y = x \otimes ay$$

Two expressions $\sum x_i \otimes y_i$ and $\sum x_i' \otimes y_i'$ are equal if and only if one of them can be rewritten into the other by using these relations.

4.10. **Flat A-modules.**

Proposition. *If M is an A-module and*

$$N' \to N \to N'' \to 0$$

is an exact sequence of A-modules, then the induced sequence

$$N' \otimes M \to N \otimes M \to N'' \otimes M \to 0$$

is exact.

In categorical language, this proposition says that the functor of modules $N \mapsto M \otimes_A N$ is **right exact**.

Example. Let $I \subseteq A$ be an ideal and M an A-module. We have a short exact sequence

$$0 \to I \to A \to A/I \to 0.$$

Tensoring with M gives an exact sequence

$$I \otimes_A M \xrightarrow{\phi} A \otimes_A M \xrightarrow{\psi} (A/I) \otimes_A M \to 0.$$

Identifying $A \otimes_A M$ and M, this becomes

$$I \otimes_A M \xrightarrow{\phi} M \xrightarrow{\psi} (A/I) \otimes_A M \to 0,$$

where the maps are defined by

$$\phi(a \otimes m) = am \quad a \in I, m \in M$$
$$\psi(m) = (1 + I) \otimes m \quad m \in M$$

Exactness means that ψ is surjective and that $\ker \psi = \mathrm{im}\, \phi$. We have $\mathrm{im}\, \phi = IM$, and then

$$(A/I) \otimes_A M = \mathrm{im}\, \psi \cong M/\ker \psi = M/\mathrm{im}\, \phi = M/IM.$$

Example. Consider the $\mathbb{Z}$-module $\mathbb{Z}/(2)$, and the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/(2) \to 0.$$

We have $\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/(2)) \cong \mathbb{Z}/(2)$, and $(\mathbb{Z}/(2)) \otimes_{\mathbb{Z}} (\mathbb{Z}/(2)) \cong \mathbb{Z}/2$. The sequence

$$\mathbb{Z} \otimes \mathbb{Z}/(2) \to \mathbb{Z} \otimes \mathbb{Z}/(2) \to \mathbb{Z}/(2) \otimes \mathbb{Z}/(2) \to 0$$

then becomes, after replacing each tensor product with its isomorphic module

$$\mathbb{Z}/(2) \xrightarrow{0} \mathbb{Z}/(2) \to \mathbb{Z}/(2) \to 0.$$

In particular, since the leftmost map is not injective, the sequence

$$0 \to \mathbb{Z}/(2) \xrightarrow{0} \mathbb{Z}/(2) \to \mathbb{Z}/(2) \to 0$$

is not exact.

To be even more concrete, note that the element $1 \otimes 1 \in \mathbb{Z} \otimes \mathbb{Z}/(2)$ is mapped to

$$2 \otimes 1 = 1 \otimes 2 = 1 \otimes 0 = 0 \otimes 0 = 0,$$

and so lies in the kernel of $\mathbb{Z} \otimes \mathbb{Z}/2 \to \mathbb{Z} \otimes \mathbb{Z}/2$.

Definition. An A-module M is **flat** if for all injective homomorphisms $N' \to N$, the induced homomorphism $M \otimes N' \to M \otimes N$ is injective.

Remark. Equivalently, M is flat if for every short exact sequence

$$0 \to N' \to N \to N'' \to 0,$$

the sequence

$$0 \to N' \otimes M \to N \otimes M \to N'' \otimes M \to 0$$

is exact.

Indeed, if M is flat, then the sequence is exact at $N' \otimes M$, and exactness at the other two modules is always true. Conversely, any injective homomorphism $\phi\colon N' \to N$ fits into a short exact sequence

$$0 \to N' \to N \to \operatorname{cok} \phi \to 0,$$

and then exactness of

$$0 \to M \otimes N' \to M \otimes N \to M \otimes \operatorname{cok} \phi \to 0,$$

implies $M \otimes N' \to M \otimes N$ is injective.

Example. The previous example shows that $\mathbb{Z}/(2)$ is not flat as a $\mathbb{Z}$-module.

Example. For any ring A, and A-module N, we have that $A \otimes N \cong N$. If $N' \to N$ is injective, then clearly $A \otimes N' \to A \otimes N$ is injective, so A is flat as an A-module.

Example. Given a collection of modules $\{M_i\}_{i \in S}$ and a module N, we have natural isomorphism

$$(\oplus_{i \in S} M_i) \otimes N \cong \oplus_{i \in S} M_i \otimes N.$$

Now if $N' \to N$ is injective and all the M_i are flat, then the morphism

$$\oplus_{i \in S} M_i \otimes N' \to \oplus_{i \in S} M_i \otimes N$$

is injective. This means $\oplus_{i \in S} M_i$ is also flat.

Example. If k is a field, then every k-module M is isomorphic to $\bigoplus_{i \in S} k$, which we know is flat. In other words, every k-module is flat.

Example. The $\mathbb{Z}$-module $\mathbb{Q}$ is flat, but we won't show this quite yet.

Algebras.

Definition. Let A be a ring. An A-**algebra** is a pair (B, ϕ), where B is a ring and $\phi\colon A \to B$ is a ring homomorphism.

If (B, ϕ) and (C, ψ) are A-algebras, then a ring homomorphism (resp. isomorphism) $\chi\colon B \to C$ is a **homomorphism (resp. isomorphism) of A-algebras** if $\chi \circ \phi = \psi$.

Remark. We usually omit the homomorphism $\phi\colon A \to B$ from the notation, and so say "B is an A-algebra" rather than "(B, ϕ) is an A-algebra". We also omit writing ϕ in relating elements of A to B, and will write things like "let $a \in A$ and $b \in B$, and consider $ab \in B$" instead of the formally correct "... consider $\phi(a)b \in B$".

Example. For any ring A, the polynomial ring $A[x_1, \ldots, x_n]$ is an A-algebra via the natural inclusion $\phi\colon A \to A[x_1, \ldots, x_n]$. All quotients $A[x_1, \ldots, x_n]/I$ are A-algebras, with A-algebra structure given by composing $A \to A[x_i] \to A[x_i]/I$.

Example. For any ring B, there is a unique ring homomorphism $\phi\colon \mathbb{Z} \to B$ given by $\phi(n) = n1_A$. Thus every ring is a $\mathbb{Z}$-algebra in precisely one way. Every homomorphism of rings is a homomorphism of $\mathbb{Z}$-algebras.

Definition. Let B be an A-algebra. We say B is a

- **finite** A-algebra if B is finitely generated as an A-module, i.e. if there exist finitely many elements $b_1, \ldots, b_n \in B$ such that every element of B can be written as
$$\sum_{i=1}^{n} a_i b_i \qquad a_i \in A, b_i \in B$$
- **finite type** A-algebra if there exist a finite set of elements $b_1, \ldots, b_n \in B$ such that every element in B can be written as a finite sum of the form
$$\sum_{i_j \geq 0} a_{i_1,\ldots,i_n} b_1^{i_1} \cdots b_n^{i_n}$$

Proposition. *Let B be an A-algebra. The following are equivalent*

(1) *B is of finite type.*
(2) *There exists a surjective A-algebra homomorphism $\phi\colon A[x_1, \ldots, x_n] \to B$*
(3) *There exists an A-algebra isomorphism $\psi\colon A[x_1, \ldots, x_n]/I \cong B$*

Proof. (1) $\implies$ (2): If B is generated by $b_1, \ldots, b_n$, then there is a surjective A-algebra homomorphism $\phi\colon A[x_1, \ldots, x_n] \to B$ given by
$$\phi\left(\sum_{i_j \geq 0} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}\right) = \sum_{i_j \geq 0} a_{i_1,\ldots,i_n} b_1^{i_1} \cdots b_n^{i_n}.$$

(2) $\implies$ (3): The fundamental homomorphism theorem gives a ring isomorphism
$$B \cong A[x_1, \ldots, x_n]/\ker\phi,$$
and it is easy to see that this is also an isomorphism of A-algebras if ϕ is an A-algebra homomorphism.

(3) $\implies$ (1): Let $\bar{x}_i = x_i + I$. Every element of $A[x_1, \ldots, x_n]/I$ can be written as a finite sum
$$\sum_{i_j \geq 0} a_{i_1,\ldots,i_n} \bar{x}_1^{i_1} \ldots \bar{x}_n^{i_n},$$