Example. Let A be an integral domain, and let $S = A \setminus \{0\}$. Then $S^{-1}A = K(A)$. To see this, observe that the equivalence relation

$$(a, s) \sim (b, t) \Leftrightarrow (as - bt)u = 0 \text{ for some } u \in S$$

used in the construction of $S^{-1}A$, and

$$(a, s) \sim (b, t) \Leftrightarrow as - bt = 0$$

used in the construction of $K(A)$, are in fact the same ones, since A is an integral domain.

More generally, if $S \subset A \setminus \{0\}$, there is a homomorphism $S^{-1}A \to K(A)$ given by $(a, s) \mapsto a/s$, identifying S^{-1} with the subring

$$\{\frac{a}{s} \mid a \in A, s \in S\} \subseteq K(A).$$

Proposition. *Let $\psi\colon A \to B$ be a ring homomorphism such that every $s \in S$, we have $\psi(s)$ is a unit in B. Then there exists a unique homomorphism $\rho\colon S^{-1}A \to B$ such that $\psi = \rho \circ \phi$.*

Proof. Uniqueness: Let $a \in A$. We must have

$$\rho(a/1) = \rho(\phi(a)) = \psi(a).$$

Let $s \in S$. We must have

$$\rho(1/s) = \rho((s/1)^{-1}) = \rho(\phi(s)^{-1}) = (\rho(\phi(s)))^{-1} = \psi(s)^{-1}.$$

Then

$$\rho(a/s) = \rho(a/1)\rho(1/s) = \psi(a)\psi(s)^{-1}.$$

Existence: Define $\rho(a/s) = \psi(a)\psi(s)^{-1}$, and check that this is well-defined. □

Example. Let A be a ring $S \subseteq A$ multiplicatively closed, and assume that $0 \in S$. Then we have, for all $a, b \in A$, $s, t \in S$, that

$$\frac{a}{s} = \frac{b}{t},$$

since

$$(ta - bs)0 = 0.$$

Thus $S^{-1}A$ has one element and is the zero ring. In particular, the homomorphism $\phi\colon A \to S^{-1}A$ is not necessarily injective.

Example. Let $S = \{1, 3\} \subseteq \mathbb{Z}/6$, and consider $S^{-1}(\mathbb{Z}/6)$. We have

$$(0, 1) \sim (2, 1)$$

since

$$(0 \cdot 1 - 2 \cdot 1) \cdot 3 = -6 = 0.$$

Similarly directly one can work out that $S^{-1}\mathbb{Z}/6$ has 2 elements, corresponding to equivalence classes

$$(0, 1), (2, 1), (4, 1), (0, 3), (2, 3), (4, 3)$$
$$(1, 1), (3, 1), (5, 1), (1, 3), (3, 3), (5, 3),$$

and so $S^{-1}(\mathbb{Z}/6) = \mathbb{Z}/2$.

Example. Let $f \in A$, and let $S = \{f^n \mid n \geq 0\} \subseteq A$. We then write $A_f = S^{-1}A$, the ring A with f inverted.

Example. Let $\mathfrak{p} \subset A$ be a prime ideal. Let $S = A \setminus \mathfrak{p}$. Then S is multiplicatively closed, and we write $A_{\mathfrak{p}} = S^{-1}A$.

Proposition. *The ring $A_{\mathfrak{p}}$ is local, with maximal ideal*

$$\mathfrak{m} = \{a/s \mid a \in \mathfrak{p}, s \notin \mathfrak{p}\}.$$

Proof. Recall a ring A is local if and only if its non-units form an ideal, and the set of non-units are then the maximal ideal of the ring.

One checks the set $\mathfrak{m}$ above is an ideal. It does not contain $1/1$, since

$$1/1 = a/s \Leftrightarrow (s-a)u = 0 \text{ for some u} \notin \mathfrak{p}.$$

If $a \in \mathfrak{p}$ and $s \notin \mathfrak{p}$, then $(s-a) \notin \mathfrak{p}$, and so $(s-a)u \notin \mathfrak{p}$, and in particular $(s-a)u \neq 0$. Therefore the elements of $\mathfrak{m}$ are non-units.

Assume next that $a/s \notin \mathfrak{m}$. Then $a \notin \mathfrak{p}$, so a/s has inverse s/a, and hence a/s is a unit. Thus $\mathfrak{m}$ is precisely the set of non-units in $A_{\mathfrak{p}}$, so $A_{\mathfrak{p}}$ is local with maximal ideal $\mathfrak{m}$. □

Example. Let $\mathfrak{p} = (p) \subseteq \mathbb{Z}$ for some prime number p. Then

$$\mathbb{Z}_{(p)} = \{m/n \mid p \text{ does not divide } n\},$$

and the maximal ideal is

$$\mathfrak{m} = \{pm/n \mid p \text{ does not divide } n\}.$$

Example. Take $\mathfrak{p} = (x) \subset \mathbb{R}[x]$. Then

$$\mathbb{R}[x]_{(x)} = \{f/g \mid g \notin (x)\},$$

the local ring from the lecture on Nakayama's lemma.

Modules on rings of fractions. Let A be a ring and let $S \subseteq A$ be a multiplicatively closed subset.

Definition. If M is an A-module, then define an $S^{-1}A$-module $S^{-1}M$ as the set of equivalence classes of pairs

$$(m, s) \qquad m \in M, s \in S,$$

under the equivalence relation

$$(m, s) \sim (n, q) \text{ if } \exists u \in S \mid (qm - sn)u = 0.$$

The module structure is

$$(m, s) + (n, q) = (qm + sn, sq)$$

and, if $a \in A$

$$(a, s)(m, q) = (am, sq).$$

Remark. We always write m/s instead of (m, s).

Example. Take $A = \mathbb{Z}$, $S = \mathbb{Z} \setminus \{0\}$, so that $S^{-1}\mathbb{Z} = \mathbb{Q}$.

Let $M = \mathbb{Z}$, then $S^{-1}M = \mathbb{Q}$.

Let $M = \mathbb{Z}/n$, then we have the following equalities in $S^{-1}M$:

$$a/s = na/ns = 0/s = 0,$$

so $S^{-1}M = 0$.

Remark. As special notational cases:

- If $S = A \setminus \mathfrak{p}$ for some prime ideal $\mathfrak{p}$, so that $S^{-1}A = A_{\mathfrak{p}}$, we write $M_{\mathfrak{p}}$ for $S^{-1}M$.
- If $S = \{1, f, f^2, \cdots$, so that $S^{-1}A = A_f$, we write M_f for $S^{-1}M$.

Remark. Given a homomorphism of A-modules $\phi\colon M \to N$, there is an induced homomorphism of $S^{-1}A$-modules $S^{-1}\phi\colon S^{-1}M \to S^{-1}N$ given by

$$S^{-1}\phi(m/s) = \phi(m)/s.$$

This means (up to verifying the two necessary conditions hold) that $M \mapsto S^{-1}M$ with the above operation on homomorphisms defines a functor from Mod_A to $\mathrm{Mod}_{S^{-1}A}$.

Proposition. *The $S^{-1}A$-module $S^{-1}M$ is isomorphic to $S^{-1}A \otimes_A M$, under the homomorphism $\phi\colon S^{-1}A \otimes_A M \to S^{-1}M$ given by*

$$a/s \otimes m \mapsto am/s$$

Proof. To see that ϕ is well-defined as a homomorphism of A-modules, check (which is easy) that the map $S^{-1}A \times M \to S^{-1}M$ given by $(a/s, m) \mapsto am/s$ is A-bilinear. The map ϕ is surjective, since for any $m/s \in S^{-1}M$ we have $\phi(1/s \otimes m) = m/s$. To see that it is injective, we claim that it has a well-defined inverse

$$\psi\colon S^{-1}M \to S^{-1}A \otimes_A M,$$

given by

$$\psi(m/s) = 1/s \otimes m.$$

Here if $m/s = n/q$, we have an $u \in S$ such that $u(qm - sn) = 0$, which implies that

$$1/s \otimes m = 1/uqs \otimes uqm = 1/uqs \otimes usn = 1/q \otimes n.$$

This means that ψ is well-defined. One checks that ϕ and ψ are homomorphisms of $S^{-1}A$-modules, and that $\psi \circ \phi$ and $\phi \circ \psi$ are both the identity maps. □

Example. In the examples above, $S^{-1}M = \mathbb{Q} \otimes_{\mathbb{Z}} M$, so our computations recover previous computations of these tensor products.

Proposition. *If $M' \xrightarrow{\phi} M \xrightarrow{\psi} M''$ is exact, then so is $S^{-1}M' \xrightarrow{S^{-1}\phi} S^{-1}M \xrightarrow{S^{-1}\psi} S^{-1}M''$.*

Proof. Let m/s be such that $S^{-1}\psi(m/s) = 0$. We must show that there exists an m'/s' such that $S^{-1}\phi(m'/s') = m/s$.

We have

$$S^{-1}\psi(m/s) = \psi(m)/s = 0/1,$$

which by definition means that there is an $u \in S$ such that $u\psi(m) = 0$.

Then $\psi(um) = 0$, and exactness implies there is an n such that $\phi(n) = um$. It follows that

$$S^{-1}\phi(n/su) = \phi(n)/su = um/su = m/s.$$

This shows that $\ker S^{-1}\psi \subseteq \operatorname{im} S^{-1}\phi$.

For the reverse containement $\operatorname{im} S^{-1}\phi \subseteq \ker S^{-1}\psi$, note that (by exactness at M) for any $m \in M'$, we have $\phi(m) \in \ker \psi$, or equivalently $\psi(\phi(m)) = 0$. It follows that for any $m/s \in S^{-1}M'$, we have $\psi(\phi(m/s)) = \psi(\phi(m))/s = 0/s = 0$, or equivalently $\phi(m/s) \in \ker \psi$. □

Corollary. *The ring $S^{-1}A$ is flat as an A-module.*

Proof. The operation $S^{-1}-$ preserves exactness, and so preserves injections, hence the operation $S^{-1}A\otimes_A -$ preserves injections, which by definition means that $S^{-1}A$ is flat. $\square$

Example. For any integral domain A, $K(A)$ is flat as an A-module.