Proposition (The operation $M \mapsto S^{-1}M$ commutes with everything)**.** *Let M be an A-module.*

- *If M' is an A-submodule of M, then $S^{-1}M'$ is an $S^{-1}A$-submodule of $S^{-1}M$, and we have $S^{-1}M/S^{-1}M' \cong S^{-1}(M/M')$.*
- *If $M', M'' \subset M$, then $S^{-1}(M' + M'') = S^{-1}M' + S^{-1}M''$, and $S^{-1}M' \cap S^{-1}M'' = S^{-1}(M' \cap M'')$.*
- *If N is an A-module, then*

$$S^{-1}M \otimes_{S^{-1}A} S^{-1}N \cong S^{-1}(M \otimes_A N).$$

Proof. Let's only prove the last one. We use the $S^{-1}A \otimes_A M = S^{-1}M$ to rewrite the left hand side as

$$\begin{aligned} S^{-1}M \otimes_{S^{-1}A} S^{-1}N &\cong M \otimes_A S^{-1}A \otimes_{S^{-1}A} S^{-1}A \otimes_A N \\ &\cong M \otimes_A S^{-1}A \otimes_A N \cong S^{-1}A \otimes_A M \otimes_A N \cong S^{-1}(M \otimes_A N). \end{aligned}$$

□

Remark. All of the above hold more generally for the operation $M \mapsto M \otimes_A B$ whenever B is a flat A-algebra.

4.12. **Local properties.** Let P be a property of module. We say (somewhat informally) that the property P is **local** if

$$\begin{gathered} P \text{ holds for } M \\ \Updownarrow \\ P \text{ holds for all localisations } M_{\mathfrak{p}}. \end{gathered}$$

Proposition ("Being 0 is local")**.** *Let M be an A-module. The following are equivalent*

(1) $M = 0$
(2) $M_{\mathfrak{p}} = 0$ *for all prime ideals* $\mathfrak{p}$
(3) $M_{\mathfrak{m}} = 0$ *for all maximal ideals* $\mathfrak{m}$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

To prove (3) $\Rightarrow$ (1), assume $M \neq 0$, and for a contradiction that $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$. Let $0 \neq m \in M$. Then

$$\operatorname{Ann}(m) = \{x \in A \mid xm = 0\} \subset A$$

is an ideal, and $\operatorname{Ann}(m) \neq (1)$. Hence there is a maximal ideal $\mathfrak{m} \supseteq \operatorname{Ann}(m)$. Now, since $M_{\mathfrak{m}} = 0$, we have

$$\frac{m}{1} = 0 \Leftrightarrow \exists u \in A \setminus \mathfrak{m} \mid um = 0,$$

but $\operatorname{Ann}(m) \subseteq \mathfrak{m}$, so this is a contradiction. □

Proposition. *Let $\phi\colon M \to N$ be a homomorphism of A-modules. Then the following are equivalent:*

(1) *ϕ is injective.*
(2) *For all prime ideals $\mathfrak{p}$, the map $\phi_{\mathfrak{p}}\colon M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is injective.*
(3) *For all maximal ideals $\mathfrak{m}$, the map $\phi_{\mathfrak{p}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective.*

Proof. The sequence $0 \to \ker \phi \to M \to N$ is exact. Since localisation is exact, we have for every prime $\mathfrak{p}$ that

$$0 \to (\ker \phi)_{\mathfrak{p}} \to M_{\mathfrak{p}} \xrightarrow{\phi_{\mathfrak{p}}} N_{\mathfrak{p}}$$

is exact. But that implies $(\ker \phi)_{\mathfrak{p}} \cong \ker \phi_{\mathfrak{p}}$.

We have ϕ injective if and only if $\ker \phi = 0$. By the above, we have $\phi_{\mathfrak{p}}$ injective if and only if $(\ker \phi)_{\mathfrak{p}} = 0$. Combining with the previous proposition gives what we want. □

Remark. The same result holds with "injective" replaced by "surjective" throughout.

Proposition. *Being flat is a local property.*

Extension and contraction of ideals in the ring of fractions. Recall that for a quotient ring A/I, we have a bijection between the set of ideals of A/I and the set of ideals of A containing I. If $\phi\colon A \to A/I$, and superscripts e and c denote extension and contraction along ϕ, the correspondence is given by

$$\mathfrak{a} \subseteq A \mapsto \mathfrak{a}^e = \mathfrak{a}/I \subseteq A/I$$

and

$$\mathfrak{a} \subseteq A/I \mapsto \mathfrak{a}^c = \phi^{-1}(\mathfrak{a}) \subseteq A.$$

A similar, but more complicated story, holds for the fractions rings $S^{-1}A$. Recall that given a multiplicatively closed subset $S \subseteq A$, we have a homomorphism $\phi\colon A \to S^{-1}A$, and we let now let superscripts c and e denote contraction and extension along this homomorphism. We begin by analysing concretely what these operations do.

Lemma. *Let $\mathfrak{a} \subset A$ be an ideal. Then*

$$\mathfrak{a}^e = \{\frac{a}{s} \mid a \in \mathfrak{a}\}$$

Proof. By definition, $\mathfrak{a}^e = \{\sum b_i/s_i\phi(a_i) \mid b_i \in A, s_i \in S, a_i \in \mathfrak{a}\}$, where we look at all finite sums. The inclusion $\supseteq$ is then clear. Conversely, $\mathfrak{a}^e$ is generated by elements of the form

$$\frac{b}{s}\phi(a) = \frac{ba}{s},$$

with $a \in \mathfrak{a}$. Then $ba \in \mathfrak{a}$, which gives the inclusion $\subseteq$. □

Lemma. *Let $\mathfrak{a} \subseteq S^{-1}A$ be an ideal. Then $\mathfrak{a}^c = \{a \in A \mid a/1 \in \mathfrak{a}\}$.*

Proof. $a \in \mathfrak{a}^c \Leftrightarrow \phi(a) \in \mathfrak{a}$ by definition, and $\phi(a) = a/1$. □

Proposition. *For any ideal $\mathfrak{a} \subset S^{-1}A$, we have $\mathfrak{a}^{ce} = \mathfrak{a}$.*

Proof. Using the lemmas above, we find

$$a/s \in \mathfrak{a} \Rightarrow (s/1)(a/s) = a/1 \in \mathfrak{a} \Rightarrow a \in \mathfrak{a}^c \Rightarrow a/s \in \mathfrak{a}^{ce},$$

and

$$a/s \in \mathfrak{a}^{ce} \Rightarrow a/s = b/q, b \in \mathfrak{a}^c \Rightarrow b/1 \in \mathfrak{a} \to (1/q)(b/1) = b/q = a/s \in \mathfrak{a}.$$

□

Corollary. *The operation $\mathfrak{a} \to \mathfrak{a}^c$ gives an inclusion of the set of ideals of $S^{-1}A$ into the set of ideals of A.*

Proof. If $\mathfrak{a}^c = \mathfrak{b}^c$, then $\mathfrak{a} = \mathfrak{a}^{ce} = \mathfrak{b}^{ce} = \mathfrak{b}$. □

Corollary. *Every ideal of $S^{-1}A$ is of the form $\mathfrak{a}^e$ for some $\mathfrak{a} \subseteq A$.*

Example. The ideals of $\mathbb{Z}_{(2)}$ are all extensions of ideals from $\mathbb{Z}$. Every ideal of $\mathbb{Z}$ is of the form (n) for some $n \geq 0$, and we have

$$(n)^e = \{\frac{nk}{s} \mid k \in \mathbb{Z}, s \notin (2)\} = (n/1)$$

If $n = 2^f q$ with q odd, then $2^f/1 = n/q \in (n/1)$, so $(2^f/1) \subseteq (n/1)$, while $n/1 = (2^f/1)(q/1) \in (2^f/1)$, so $(n/1) \subseteq (2^f/1)$. It follows that $(n/1) = (2^f/1)$. Thus the complete set of ideals in $\mathbb{Z}_{(2)}$ is

$$(1) \supsetneq (2/1) \supsetneq (2^2/1) \supsetneq \cdots$$

and (0).