Let's focus now on the case of prime ideals.

Proposition. *Let $S \subseteq A$ be a multiplicative subset, and let $\phi\colon A \to S^{-1}A$ be the standard homomorphism. There is a bijection*

$$\{\textit{prime ideals of } S^{-1}A\} \leftrightarrow \{\textit{prime ideals } \mathfrak{q} \textit{ of } A \textit{ with } \mathfrak{q} \cap S = \varnothing\}$$
$$\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$$
$$\phi(\mathfrak{q})S^{-1}A \hookleftarrow \mathfrak{q}$$

Proof. Let $\mathfrak{p} \subset S^{-1}A$ be a prime ideal. Then $\phi^{-1}(\mathfrak{p}) \subset A$ is a prime ideal (*Contraction always preserves prime ideals*). Moreover, $\phi^{-1}(\mathfrak{p}) \cap S = \varnothing$, since if $a \in \phi^{-1}(\mathfrak{p}) \cap S$, then

$$\phi(a) = \frac{a}{1} \in \mathfrak{p} \overset{a \in S}{\Longrightarrow} \frac{1}{a}\frac{a}{a} = 1 \in \mathfrak{p} \Rightarrow \mathfrak{p} = (1),$$

contradicting primality of $\mathfrak{p}$.

If $\mathfrak{q} \subset A$ is a prime ideal with $\mathfrak{q} \cap S = \varnothing$, then we claim (1) $\mathfrak{q}S^{-1}A$ is a prime ideal, and (2) $\phi^{-1}(\mathfrak{q}S^{-1}A) = \mathfrak{q}$.

For (1), assume for a contradiction that $\mathfrak{q}S^{-1}A$ is not a prime ideal. Then we can find $a/s, b/q \in S^{-1}A$ such that $a, b \notin \mathfrak{q}$ with

$$\frac{a}{s}\frac{b}{q} = \frac{ab}{qs} = \frac{c}{t},$$

with $c \in \mathfrak{q}$, $t \in \mathfrak{S}$. That implies $(abt - qsc)u = 0$ for some $u \in S$, or equivalently $abtu = qscu$. But $a, b, t, u \notin \mathfrak{q}$, so $abtu \notin \mathfrak{q}$. On the other hand $c \in \mathfrak{q}$ implies that $qscu \in \mathfrak{q}$, so we have a contradiction.

For (2), we always have $\mathfrak{q} \subseteq \phi^{-1}(\mathfrak{q}S^{-1}A)$, so we only need to prove that $\phi^{-1}(\mathfrak{q}S^{-1}A) \subseteq \mathfrak{q}$. If $a \in \phi^{-1}(\mathfrak{q}S^{-1}A)$, then

$$\phi(a) = a/1 = b/s$$

with $b \in \mathfrak{q}$ and $s \in S$. This implies that $(as - b)u = 0$ for some $u \in S$, or equivalently $as = bu$. Since $b \in \mathfrak{q}$, we have $bu \in \mathfrak{q}$, and since $s \notin \mathfrak{q}$ we must then have $a \in \mathfrak{q}$. Hence $\phi^{-1}(\mathfrak{q}S^{-1}A) \subseteq \mathfrak{q}$.

We have shown the two operations in the statement of the theorem are mutually inverse, so they give a bijection. □

Corollary. *Let $\mathfrak{p} \subset A$ be a prime ideal. The prime ideals of $A_{\mathfrak{p}}$ are precisely the ideals $\mathfrak{q}A_{\mathfrak{p}}$, where $\mathfrak{q} \subseteq \mathfrak{p}$ is a prime ideal.*

Proof. The condition $\mathfrak{q} \cap (S \setminus \mathfrak{p}) = \varnothing$ is equivalent to $\mathfrak{q} \subseteq \mathfrak{p}$. □

Example. The prime ideals of $\mathbb{Z}_{(2)}$ are exactly $(2)\mathbb{Z}_{(2)} = (2/1)$ and $(0)\mathbb{Z}_{(2)} = (0/1)$.

Note that the bijection proposition above fails here for non-prime ideals, i.e. $(6) \subset (2) \subset \mathbb{Z}$, but $(6)\mathbb{Z}_{(2)} = (2)\mathbb{Z}_{(2)} = (2/1)$, and there is no ideal $I \subseteq \mathbb{Z}_{(2)}$ such that $\phi^{-1}(I) = (6)$.

Corollary. *Let $f \in A$. The prime ideals of A_f are precisely the ideals $\mathfrak{q}A_f$, where $\mathfrak{q} \subset A$ is a prime ideal not containing f.*

Proof. Since $A_f = S^{-1}A$ with $S = \{f^k\}_{k \geq 0}$, the prime ideals of A_f are $\mathfrak{q}A_f$ where $\mathfrak{q} \cap S = \varnothing$. Now $f \in \mathfrak{q} \Leftrightarrow f^k \in \mathfrak{q}$ for some $k \geq 0$, since $\mathfrak{q}$ is a prime ideal. Hence $S \cap \mathfrak{q} = \varnothing \Leftrightarrow f \notin \mathfrak{q}$. □

Example. The prime ideals of $\mathbb{Z}_2$ are $(3/1), (5/1), (7/1), (11/1), \ldots$.

We can give better proofs of a few things we've seen before.

Corollary. *The ring $A_{\mathfrak{p}}$ is local.*

Proof. Every prime ideal of $A_{\mathfrak{p}}$ is of the form $\mathfrak{q}A_{\mathfrak{p}}$ for some $\mathfrak{q} \subseteq \mathfrak{p}$. Hence every prime ideal of $A_{\mathfrak{p}}$ is contained in $\mathfrak{p}A_{\mathfrak{p}}$, which must then be the unique maximal ideal. □

Proposition. *The nilradical of A is the intersection of all the prime ideals of A.*

Proof. For $f \in A$, we must show that the condtion "f is nilpotent" is equivalent to "f lies in every prime ideal".

If f is nilpotent, then we have $f^n = 0$ for some $n \geq 0$. For every prime ideal $\mathfrak{p}$, then $f^n \in \mathfrak{p}$, and so $f \in \mathfrak{p}$.

If f lies in every prime ideal, then the set of prime ideals $\mathfrak{p}$ such that $f \notin \mathfrak{p}$ is empty. It follows that the ring A_f has no prime ideals, which means that it is the 0 ring. This means that in A_f we have the equality

$$\frac{1}{1} = \frac{0}{1},$$

which in turn means that there exists some $n \geq 0$ such that $f^n(1 \cdot 0 - 0 \cdot 0) = 0$, i.e. that f is nilpotent. □