## Chain conditions II

**Proposition.** *Let $M$ be an $A$-module. Then $M$ is Noetherian if and only if every submodule of $M$ is finitely generated.*

*Proof.* Assume that $M$ is Noetherian, and let $M' \subseteq M$ be a submodule. Let

$$T = \{N \subseteq M' \mid N \text{ a finitely generated submodule of } M'\}.$$

By the Noetherian hypothesis, there is a maximal element $N_{max} \in T$. Assume for a contradiction that $N_{max} \neq M'$. Then there is an $m \in M' \setminus N_{max}$, so

$$N_{max} \subsetneq N_{max} + Am \subseteq M'.$$

If $N_{max}$ is generated by $x_1, \ldots, x_r$, then $N_{max} + Am$ is generated by $x_1, \ldots, x_r, m$; in particular $N_{max} + Am$ is finitely generated. But then $N_{max} + Am \in T$, contradicting the maximality of $N_{max}$. Thus we must have $N_{max} = M'$, so $M'$ is finitely generated.

Assume that every submodule $M' \subseteq M$ is finitely generated. Let

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

be a chain of submodules, and let

$$M' = \bigcup_{i=1}^{\infty} M_i \subseteq M.$$

Then $M'$ is by assumption finitely generated, say by $m_1, \ldots, m_n$. For each $i = 1, \ldots, n$, there is a $k_i$ such that $m_i \in M_{k_i}$. Taking $k = \max(k_1, \ldots, k_n)$, we have $m_1, \ldots, m_n \in M_k$. Since $M_k$ contains every generator of $M'$, we have $M' \subseteq M_k$, and by definition of $M'$ we have $M_k \subseteq M'$, so $M_k = M'$. For every $n \geq k$, we have $M_k \subseteq M_n \subseteq M'$, so $M_k = M_n$, which means the chain stabilises at $M_k$. □

**Example.** A ring $A$ is Noetherian if and only if every ideal is finitely generated. Thus if $k$ is a field, then $k[x]$ is Noetherian since every ideal of $k[x]$ is principal.

**Theorem.** *Let*

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$$

*be a short exact sequence of $A$-modules. Then*

$$M \text{ Noetherian} \Leftrightarrow M' \text{ and } M'' \text{ Noetherian.}$$

*and*

$$M \text{ Artinian} \Leftrightarrow M' \text{ and } M'' \text{ Artinian.}$$

*Proof.* We only do the statement for the Noetherian condition.

$\Rightarrow$: If

$$M_1' \subseteq M_2' \subseteq \cdots$$

is a chain of submodules of $M'$, then

$$i(M_1') \subseteq i(M_2') \subseteq \cdots .$$

is a chain of submodules of $M$. Since $M$ is Noetherian, the second chain stabilises, so the first one must as well.

If

$$M_1' \subseteq M_2' \subseteq \cdots$$

is a chain of submodules of $M''$, then

$$p^{-1}(M_1') \subseteq p^{-1}(M_2') \subseteq \cdots .$$

is a chain of submodules of $M$. Since $M$ is Noetherian, the latter stabilises, so the first one must as well.

$\Leftarrow$: If

$$M_1 \subseteq M_2 \subseteq \cdots$$

is a chain of submodules of $M$, then we get chains

$$i^{-1}(M_1) \subseteq i^{-1}(M_2) \subseteq \cdots$$

and

$$p(M_1) \subseteq p(M_2) \subseteq \cdots .$$

By the Noetherian assumption, both of these stabilise, so for some $N$ we have that for all $i \geq N$, then

$$i^{-1}(M_i) = i^{-1}(M_{i+1})$$

and

$$p(M_i) = p(M_{i+1}).$$

*Claim:* It follows that $M_i = M_{i+1}$. It is not hard to prove this directly,[6] but it is a good excuse to show off the snake lemma:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & i^{-1}(M_i) & \xrightarrow{i} & M_i & \xrightarrow{p} & p(M_i) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & i^{-1}(M_{i+1}) & \xrightarrow{i} & M_{i+1} & \xrightarrow{p} & p(M_{i+1}) & \longrightarrow & 0. \end{array}$$

The snake lemma gives an exact sequence

$$0 \to \ker f' \to \ker f \to \ker f'' \to \operatorname{cok} f' \to \operatorname{cok} f \to \operatorname{cok} f'' \to 0,$$

and since $\operatorname{cok} f' = \operatorname{cok} f'' = 0$, we get that $\operatorname{cok} f = 0$, so $f$ is surjective. We've shown that the sequence $M_i$ stabilises. □

**Corollary.** *If $M_1, \ldots, M_n$ are Noetherian (resp. Artinian) $A$-modules, then so is*

$$\bigoplus_{i=1}^{n} M_i.$$

*Proof.* Inductively prove that $\oplus_{i=1}^{j} M_i$ is Noetherian, using the exact sequence

$$\begin{array}{ccccccc} 0 & \to & M_{j+1} & \to & \displaystyle\bigoplus_{i=1}^{j+1} M_i & \to & \displaystyle\bigoplus_{i=1}^{j} M_i \to 0 \\ & & x & \mapsto & (0, \ldots, 0, x) & & \\ & & & & (x_1, \ldots, x_{j+1}) & \mapsto & (x_1, \ldots, x_j) \end{array}$$

□

**Proposition.** *Let $A$ be a Noetherian (resp. Artinian) ring, and let $M$ be a finitely generated $A$-module. Then $M$ is Noetherian (resp. Artinian).*

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[6] If $m \in M_{i+1}$, then $p(m) \in p(M_{i+1}) = p(M_i)$, so there is some $m' \in M_{i+1}$ such that $p(m) = p(m')$. But then $p(m - m') = 0$, so there is some $m'' \in M'_{i+1}$ such that $i(m'') = m - m'$. Since $i^{-1}M_{i+1} = i^{-1}M_i$, we have that $m'' \in M'_i$, and therefore $m = m' + i(m'') \in M_i$.

*Proof.* $A$ is Noetherian $\Rightarrow A^n$ is Noetherian. There is some surjective homomorphism $\phi\colon A^n \to M$, and the short exact sequence

$$0 \to \ker\phi \to A^n \to M \to 0$$

shows that $M$ is Noetherian. □

## Simple modules and composition series

**Definition.** A module $M$ is **simple** if it has no proper nontrivial submodules.

**Example.** If $A$ is a ring with a maximal ideal $\mathfrak{m} \subset A$, then $A/\mathfrak{m}$ is a simple $A$-module: If $0 \subseteq M \subseteq A/\mathfrak{m}$ is a chain of modules, and $p\colon A \to A/\mathfrak{m}$ is the projection, then

$$p^{-1}(0) = \mathfrak{m} \subseteq p^{-1}(M) \subseteq p^{-1}(A/\mathfrak{m}) = A$$

is a chain of submodules (ideals) of $A$. Since $\mathfrak{m}$ is maximal, then either $p^{-1}(M) = \mathfrak{m}$ or $p^{-1}(M) = A$, which implies $M = 0$ or $M = A/\mathfrak{m}$.

**Proposition.** *Every simple $A$-module $M$ is isomorphic to $A/\mathfrak{m}$ for a maximal ideal $\mathfrak{m}$.*

*Proof.* Let $M$ be simple, and let $0 \neq m \in M$. There is a homomorphism $\phi\colon A \to M$ defined by $\phi(a) = am$. Since $m \neq 0$, we have $0 \neq \operatorname{im}\phi \subseteq M$, and so since $M$ is simple we must have $M = \operatorname{im}\phi$. Thus $M \cong A/\ker\phi$. We claim the ideal $\ker\phi$ is maximal, suppose for a contradiction that it is not. There is then a maximal ideal $\mathfrak{m} \supsetneq \ker\phi$, so we get $0 \neq \mathfrak{m}/\ker\phi \subsetneq A/\ker\phi$, which is a contradiction since $A/\ker\phi \cong M$ is simple. □

**Definition.** A **composition series** of a module $M$ is a finite chain

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{n-1} \supsetneq M_n = 0,$$

which is maximal, that is it cannot be extended to a longer chain by adding some submodule $M'$ with

$$M_i \supsetneq M' \supsetneq M_{i+1}.$$

**Proposition.** *A chain*

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$$

*is a composition series if and only if $M_i/M_{i+1}$ is simple for every $i$.*

*Proof.* Suppose there is an $M'$ such that $M_i \supsetneq M' \supsetneq M_{i+1}$. Then $0 \neq M'/M_{i+1} \subsetneq M_i/M_{i+1}$, so $M_i/M_{i+1}$ is not simple. Conversely, supposse that $M_i/M_{i+1}$ is not simple. Then there is some $0 \neq N \subsetneq M_i/M_{i+1}$. If $\phi\colon M_i \to M_i/M_{i+1}$ is the quotient map, then we have

$$M_{i+1} \subsetneq \phi^{-1}(N) \subsetneq M_i$$

□

**Example.** Let $p$ be a prime, $k \geq 1$, and consider the $\mathbb{Z}$-module $\mathbb{Z}/(p^k)$. This has a composition series of length $k$, given by

$$\mathbb{Z}/(p^k) \supsetneq (p)/(p^k) \supsetneq (p^2)/(p^k) \supsetneq \cdots \supsetneq (p^k)/(p^k) = 0.$$

The quotients are $((p^i)/(p^k))/(p^{i+1})/(p^k) \cong (p^i)/(p^{i+1}) \cong \mathbb{Z}/p$, so are simple.

**Example.** Let $p$ and $q$ be primes, and consider the module $\mathbb{Z}/(pq)$. This has two compositions series

$$\mathbb{Z}/(pq) \supsetneq (p)/(pq) \supsetneq (pq)/(pq) = 0$$

and

$$\mathbb{Z}/(pq) \supsetneq (q)/(pq) \supsetneq (pq)/(pq) = 0$$

**Example.** Let $k$ be a field. Then $k$ is the only simple $k$-module. Thus if a $k$-module $M$ has a composition series

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0,$$

then $M_i/M_{i-1} \cong k$. It follows that $\dim M_i = \dim M_{i-1} + 1$, and by induction we get that $\dim M_i = n - i$. In particular $n = \dim M$.

**Definition.** The **length** of a module $M$, denoted $l(M)$, is the minimal length of a composition series of $M$ and $\infty$ if $M$ admits no composition series.

**Lemma.** *Let $M$ be a module with $l(M) < \infty$. If $N$ is a proper submodule of $M$, then $l(N) < l(M)$.*

*Proof.* Let

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{l(M)} = 0$$

be a composition series of minimal length. We claim

$$N = M_0 \cap N \supseteq M_1 \cap N \supseteq \cdots \supseteq M_{l(M)} \cap N = 0$$

contains a composition series of $N$, in the sense that we can find

$$0 = j_0 < j_1 < \cdots < j_k \leq l(M)$$

such that

$$N = M_{j_0} \cap N \supsetneq M_{j_1} \cap N \supsetneq \cdots \supsetneq M_{j_k} \cap N = 0$$

is a composition series. For every $i$, we have a homomorphism

$$\phi \colon M_i \cap N \hookrightarrow M_i \to M_i/M_{i+1},$$

with

$$\ker \phi = M_{i+1} \cap N.$$

and with

$$\operatorname{im} \phi = M_i/M_{i+1} \text{ or } \operatorname{im} \phi = 0,$$

since $M_i/M_{i+1}$ is simple. Hence

$$(M_i \cap N)/(M_{i+1} \cap N) = (M_i \cap N)/\ker \phi \cong \operatorname{im} \phi \begin{cases} M_i/M_{i+1}(\text{ Case 1}) \\ 0(\text{Case 2}). \end{cases}$$

In Case 1, we have

$$N \cap M_i = N \cap M_{i+1}$$

and in Case 2,

$$(N \cap M_i)/(N \cap M_{i+1})$$

is simple. Taking the sequence

$$N \cap M_0 \supsetneq N \cap M_{j_1} \supsetneq N \cap M_{j_2} \supsetneq \cdots \supsetneq N \cap M_{j_k} = 0,$$

where $0 = j_0 < j_1 < \cdots < j_k \leq l(m)$ are the indices such that $N \cap M_{j_k} \neq N \cap M_{j_k - 1}$, we have produced a composition series of $N$ of length $k \leq l(M)$, proving $l(N) \leq l(M)$.

Now as $N \subsetneq M$, we have

$$N = N \cap M_0 \neq M_0 = M,$$

while

$$0 = N \cap M_{l(M)} = M_{l(M)} = 0.$$

Let $i > 0$ be the smallest number such that $N \cap M_i = M_i$. Then we have

$$M_{i-1} \supsetneq N \cap M_{i-1} \supseteq N \cap M_i = M_i,$$

which shows that $N \cap M_{i-1} = N \cap M_i$, so $i$ is not in the set $\{j_l\}_{l=1}^{k}$ Hence $k = l(N) < l(M)$. □

**Proposition.** *Let $M$ be a module with $l(M) = n$. Then*

(1) *every composition series of $M$ has length $n$*
(2) *for every chain*

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_k = 0$$

*we have $k < n$, and we can extend the chain to a composition series by adding $(n - k$ many) modules $M'$ with $M_i \supsetneq M' \supsetneq M_{i+1}$.*

*Proof.* (1): Let

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_k = 0$$

be a composition series. By the lemma, we have

$$l(M) = l(M_0) > l(M_1) > \cdots > l(M_k) = 0,$$

so $l(M) \geq k$. But by definition $l(M) \leq k$, so $l(M) = k$.

(2): If

$$M = M_0 \supsetneq \cdots \supsetneq M_k = 0$$

is a chain of length $k < l(M)$, then by definition of $l(M)$ it cannot be a composition series, so we can add some $M'$ with $M_i \supsetneq M' \supsetneq M_{i+1}$. □

**Remark.** Worth knowing, but not something we will prove or focus on, is the **Jordan–Hölder theorem**. This says that given two composition series $M_i$ and $M_i'$ of a module $M$, the isomorphism classes of modules appearing in $\{M_i/M_{i+1}\}_{i=1}^{l(M)}$ and $\{M_i'/M_{i+1}'\}_{i=1}^{l(M)}$ are the same. A given isomorphism class appears the same number of times in each of the two sets.

**Example.** Given distinct primes $p$ and $q$, the two composition series for $\mathbb{Z}/(pq)$ are

$$\mathbb{Z}/(pq) \supsetneq (p)/(pq) \supsetneq 0$$

and

$$\mathbb{Z}/(pq) \supsetneq (q)/(pq) \supsetneq 0.$$

We have

$$(\mathbb{Z}/(pq))/((p)/(pq)) \cong \mathbb{Z}/p, \quad ((p)/(pq))/0 \cong \mathbb{Z}/q$$

and

$$(\mathbb{Z}/(pq))/((q)/(pq)) \cong \mathbb{Z}/q, \quad ((q)/(pq))/0 \cong \mathbb{Z}/p.$$

**Proposition.** *Let $M$ be a module. Then $M$ has finite length $\Leftrightarrow$ $M$ is Noetherian and Artinian.*

*Proof.* $\Rightarrow$: Any increasing sequence has at most $l(M)$ distinct terms, similarly for a decreasing sequence.

$\Leftarrow$: Define a descending chain as follows: Let $M_0 = M$, and let $M_1$ be a maximal submodule of $M$ not equal to $M$. This exists because $M$ is Noetherian. Inductively define $M_{i+1}$ as a maximal submodule of $M_i$ among those not equal to $M_i$. The sequence $M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots$ cannot be extended indefinitely, since $M$ is Artinian, hence we eventually have $M_n = 0$. Then $M_i$ defines a composition series for $M$. □

**Proposition.** *Given a short exact sequence of modules*

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0,$$

*we have* $l(M) = l(M') + l(M'')$.

*Proof.* The case where one of $l(M), l(M')$ or $l(M'')$ is $\infty$ can be handled by the previous proposition and the fact that $M$ is Noetherian (resp. Artinian) if and only if $M'$ and $M''$ are.

Hence we can assume that $M'$, $M$ and $M''$ are all of finite length. Take a composition series $M'_i$ for $M'$ and $M''_j$ for $M''$. These induce the following sequence of submodules of $M$:

$$\begin{aligned} M &= p^{-1}(M''_0) \supsetneq p^{-1}(M''_1) \supsetneq \cdots p^{-1}(M''_{l(M'')}) = p^{-1}(0) = i(M') \\ &= i(M'_0) \supsetneq i(M'_1) \supsetneq \cdots i(M'_{l(M')}) = 0. \end{aligned}$$

Since

$$p^{-1}(M''_i)/p^{-1}(M''_{i+1}) \cong M''_i/M''_{i+1}$$

and

$$i(M'_i)/i(M'_{i+1}) \cong M'_i/M'_{i+1}$$

this gives a composition series of length $l(M') + l(M'')$ for $M$. □