Noetherian rings

Recall a ring A is Noetherian if either of the following equivalent conditions hold

(1) Every ascending chain of ideals stabilises.
(2) Every set of ideals has a maximal element.
(3) Every set of ideals is finitely generated.

Proposition. *Let A be a Noetherian (resp. Artinian) ring.*

(1) *Let $S \subseteq A$ be a multiplicatively closed subset. The ring S^{-1} is Noetherian (resp. Artinian).*
(2) *Let $\mathfrak{a} \subseteq A$ be an ideal. The ring $A/\mathfrak{a}$ is Noetherian (resp. Artinian).*

Proof. We only do the Noetherian case, the Artinian is similar.

(1): Let $\phi\colon A \to S^{-1}A$ be the usual homomorphism. We know that the map $\mathfrak{b} \mapsto \phi^{-1}(\mathfrak{b})$ is an injective and order-preserving map from the set of ideals of $S^{-1}A$ to the set of ideals of A. An ascending chain

$$\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \cdots$$

gives an ascending chain

$$\phi^{-1}\mathfrak{b}_1 \subseteq \phi^{-1}\mathfrak{b}_2 \subseteq \cdots$$

Since A is Noetherian, the latter stabilises, hence the former stabilises.

(2): Let $\psi\colon A \to A/\mathfrak{a}$ be the quotient homomorphism. Again $\mathfrak{b} \mapsto \psi^{-1}\mathfrak{b}$ is an injective and order-preserving map from the set of ideals of $A/\mathfrak{a}$ to the set of ideals of A, and we can conclude similarly to case (1). □

Theorem (Hilbert's basis theorem)**.** *Let A be a Noetherian ring. Then $A[x]$ is Noetherian.*

Proof. Let $\mathfrak{a} \subseteq A[x]$. Our goal is to show that $\mathfrak{a}$ is finitely generated.

Define

$$\mathfrak{a}_n = \{a_n \in A \mid a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathfrak{a}\}.$$

In words, $\mathfrak{a}_n$ is the set of leading terms of degree n polynomials in $\mathfrak{a}$.

Easy claim 1: $\mathfrak{a}_n \subseteq A$ is an ideal.

Easy claim 2: $\mathfrak{a}_n \subseteq \mathfrak{a}_{n+1}$ for every n.

Since A is Noetherian, there is an N such that the chain of $\mathfrak{a}_i$ stabilises at $\mathfrak{a}_N$, i.e. we have

$$\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \cdots \subseteq \mathfrak{a}_N = \mathfrak{a}_{N+1} = \cdots .$$

Now for each $i = 0, \ldots, N$, we can find a finite set of generators $a_{i,j} \in A$ for $\mathfrak{a}_i$, so that e.g.

$$(a_{i,1}, \ldots, a_{i,k_i}) = \mathfrak{a}_i$$

For each i, j, the fact that $a_{i,j} \in \mathfrak{a}_i$ means there is an $f_{i,j} \in \mathfrak{a}$ such that

$$f_{i,j} = a_{i,j}x^i + \text{lower order terms.}$$

Main claim: The ideal $\mathfrak{a}$ is generated by these $f_{i,j}$. *Proof:* Assume for a contradiction that they do not, in which case there is some $g \in \mathfrak{a}$ which does not lie in the ideal $(f_{i,j})_{i,j}$. We may assume that the degree of g is minimal with this property. There are two cases:

- $\deg g < N$: Writing
$$g = a_i x^i + \text{lower order terms},$$
we have $a_i \in \mathfrak{a}_i = (a_{i,j})$. We can then write
$$a_i = \sum_{j=1}^{k_i} c_i a_{i,j}, \qquad c_i \in A.$$
Considering
$$g' = g - \sum_{j=1}^{k_i} c_i f_{i,j} = (a_i - \sum c_i a_{i,j}) x^i + \text{ lower order terms},$$
we have $\deg g' < \deg g$, and clearly $g' \in \mathfrak{a}$. By induction on degree $g' \in (f_{i,j})$, so $g \in (f_{i,j})$.
- If $\deg g \geq N$, take instead
$$g' = g - \sum_{j=1}^{r_N} c_i f_{N,j} x^{\deg g - N},$$
and conclude similarly.

□

Example. Consider the case of a field k, where the proof above simplifies as follows. For any ideal $\mathfrak{a} \subseteq k[x]$, we have $\mathfrak{a}_i = (0)$ or $\mathfrak{a}_i = (1)$. We thus get
$$0 = \mathfrak{a}_0 = \mathfrak{a}_1 = \cdots = \mathfrak{a}_{N-1} \subsetneq \mathfrak{a}_N = (1) = \mathfrak{a}_{N+1} = \cdots .$$
There is some $f_{N,1} \in \mathfrak{a}_N$ such that
$$f_{N,1} = x^N + \text{ lower order terms}.$$
Then $\mathfrak{a} = (f_{N,1})$.

Corollary. *If A is a Noetherian ring, then so is $A[x_1, \ldots, x_n]$.*

Proof. A Noetherian $\Rightarrow$ $A[x_1]$ Noetherian $\Rightarrow$ $A[x_1, x_2] \cong A[x_1][x_2]$ Noetherian and so on. □

Corollary. *If A is Noetherian and B is an A-algebra of finite type, then B is Noetherian.*

Proof. B is of finite type if it is isomorphic (as A-algebra) to $A[x_1, \ldots, x_n]/\mathfrak{a}$. Now A Noetherian $\Rightarrow$ $A[x_1, \ldots, x_n]$ Noetherian $\Rightarrow$ $A[x_1, \ldots, x_n]/\mathfrak{a}$ Noetherian. □

6. Hilbert's (weak) nullstellensatz

Recall that a field extension $k \subseteq k'$ is finite if k' is a finite-dimensional k-vector space.

Proposition (Zariski's lemma)**.** *Let k be a field, and let A be a finitely generated k-algebra. If A is a field, then A is a finite field extension of k.*

Remark. Clearly, if A is a finite field extension of k, then it is finitely generated as a k-algebra, since then A has a k-basis $a_1, \ldots, a_n$ which generates A as a module over k. These must therefore also generate A as a k-algebra.

Example. The field extension $k \subset k(x)$ is not finite, so the lemma says in this case that $k(x)$ is not a finitely generated k-algebra. Concretely, the this means it is not possible to find rational functions $f_1, \ldots, f_n \in k(x)$ such that every rational function $f \in k(x)$ can be written as

$$f = \sum_{i=1}^{n} g_i f_i, \qquad g_i \in k[x]$$

Corollary (Weak nullstellensatz)**.** *Let $\mathfrak{m} \subseteq k[x_1, \ldots, x_n]$ be a maximal ideal. Then $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a finite field extension of k.*

If k is algebraically closed, then $k[x_1, \ldots, x_n]/\mathfrak{m} \cong k$, and $\mathfrak{m}$ has the form

$$\mathfrak{m} = (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$$

for some $a_1, \ldots, a_n \in k$.

Proof. The ring $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a field, so Zariski's lemma applies.

If k is algebraically closed, then for a finite field extension $k \subseteq k'$ we must have $k = k'$. Thus the homomorphism $k \to k[x_1, \ldots, x_n]/\mathfrak{m}$ must be an isomorphism. In the ring $k[x_1, \ldots, x_n]/\mathfrak{m}$, for each $i = 1, \ldots, n$, we have $x_i = a_i$ for some $a_i \in k$. This means that $x_i - a_i \in \mathfrak{m}$, and hence

$$(x_1 - a_1, \ldots, x_n - a_n) \subseteq \mathfrak{m}.$$

But it is easy to see that the ideal on the left hand side is maximal, so we have an equality. □

Proof of Zariski's lemma, cheap version. We will assume that k is uncountable, (e.g. $k = \mathbb{C}, \mathbb{R}$, not $k = \mathbb{F}_p, \mathbb{Q}, \overline{\mathbb{Q}}$). The lemma holds without this assumption, but the proof is more involved.

Let A be a k-algebra generated by $a_1, \ldots, a_n \in A$. Then A is generated as a k-module by the elements $a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n}$, of which there are countably many, so that A has countable dimension as a k-module, i.e.

$$\dim_k A \leq |\mathbb{N}|.$$

We claim that each of the a_i are algebraic over k, i.e. that for each i there is some non-zero $f \in k[x]$ such that $f(a_i) = 0$. Assume for a contradiction that a_i is not algebraic over k, i.e. that $f(a_i) \neq 0$ for all $0 \neq f \in k[x]$. We then have a homomorphism of fields

$$\phi \colon k(x) \to A$$
$$\phi(f/g) = f(a_i)/g(a_i).$$

Since $k(x)$ is a field, ϕ is injective, and since $k(x) \cong \phi(k(x))$ is a k-submodule of A, we then have

$$\dim_k k(x) = \dim_k \phi(k(x)) \leq \dim_k A$$

Claim: The dimension of $k(x)$ as a k-module is at least $|k|$.

Proof. For each $\alpha \in k$, we have an element $(x - \alpha)^{-1} \in k(x)$. We claim these are all linearly indendent. Suppose we have a linear relation

$$\sum_{i=1}^{n} a_i (x - \alpha_i)^{-1} = 0 \qquad b_i \in k$$

with the α_i pairwise distinct. Multiplying by $f = \prod_{i=1}^{n}(x - \alpha_i)$ gives

$$0 = \sum_{i=1}^{n} a_i \frac{f}{(x - \alpha_i)} \in k[x].$$

Evaluating this polynomial in α_i gives

$$0 = a_i \prod_{j \neq i} (\alpha_j - \alpha_i),$$

and so $a_i = 0$. It follows that the linear relation was trivial, which means that the $(x - \alpha_i)^{-1}$ are linearly independent over k. The set $S = \{(x - \alpha)^{-1}\}_{\alpha \in k} \subset k(x)$ can be extended to a basis S' of $k(x)$ as k-module, and so

$$\dim_k k(x) = |S'| \geq |S| = |k|.$$

□

We have now found that

$$|k| \leq \dim_k k(x) \leq \dim_k A \leq |\mathbb{N}|,$$

which is false since k was assumed uncountable. This means our assumption that a_i was not algebraic over k was incorrect. Hence each $a_1, \ldots, a_n$ are algebraic over k, and thus A is finite as a k-module. □