Graded rings and Hilbert polynomials

Definition. A **graded ring** is a ring A together with subgroups $A_i \subseteq A$ for each $i \geq 0$ such that

$$A = \bigoplus_{i=0}^{\infty} A_i,$$

and such that for every $i, j \geq 0$, we have

$$a \in A_i, b \in A_j \Rightarrow ab \in A_{i+j}.$$

Remark. The condition that $A = \bigoplus_{i=0}^{\infty} A_i$ is equivalent to requiring that for every $a \in A$, we can write

$$a = \sum_{i=0}^{\infty} a_i \qquad a_i \in A_i$$

in a unique way (with only finitely many $a_i \neq 0$).

Example. For any ring A, the ring $A[x]$ is graded, by setting

$$A[x]_i = \{ax^i \mid a \in A\} \subset A[x]$$

More generally, the ring $A[x_1, \ldots, x_n]$ is graded by setting

$$f \in A[x_1, \ldots, x_n]_i$$

if and only if f is a sum of terms of the form $ax_1^{i_1} \cdots x_n^{i_n}$ with $a \in k$ and $\sum_{k=1}^{n} i_k = i$.

Definition. If A is a graded ring and $a \in A_i$, we say that a is **homogeneous of degree** i.

Remark. Every ring A can be a made into a graded ring in a "trivial" way by setting $A_0 = A$, and $A_i = 0$ for $i \geq 1$.

Remark. A ring A can potentially be made into a graded ring in many different ways: For any choice of $d, e \geq 0$, we can define a grading on the ring $k[x, y]$ by setting

$$A_i = \{\sum a_{j,k} x^j y^k \mid dj + ek = i\}.$$

Lemma. *The group $A_0 \subseteq A$ is a subring of A.*

Proof. By assumption, A_0 is an additive subgroup of A which is closed under multiplication, so it remains to check that $1 \in A_0$. We may write $1 = \sum_{i=0}^{n} x_i$ with each x_i homogeneous of degree i. For every $j = 0, \ldots, n$, we have

$$x_j = 1x_j = \left(\sum_{i=0}^{n} x_i\right) x_j = \sum_{i=0}^{n} x_i x_j.$$

Since $x_i x_j$ is homogeneous of degree $i + j$, we then get

$$x_j = x_0 x_j \text{ and } x_i x_j = 0 \text{ for } i \geq 1.$$

It follows that

$$x_0 = 1x_0 = \left(\sum_{i=0}^{n} x_i\right) x_0 = \sum_{i=0}^{n} x_i = 1.$$

□

Example. If $f \in A$ is a homogeneous element, then $B = A/(f)$ is also a graded ring. If $\phi\colon A \to A/(f)$ is the quotient map, we have

$$B_i = \phi(A_i) \subseteq B.$$

It is an easy exercise to check that this defines a valid grading of B, but note that it is necessary that f is homogeneous. More generally, if $f_1, \ldots, f_n \in A$ are homogeneous, the ring $A/(f_1, \ldots, f_n)$ inherits a grading from A.

Assumption: We will for the rest of these lectures assume of our graded ring A that $A_0 = k$ is a field, and that A is generated as a k-algebra by homogeneous elements $x_1, \ldots, x_n$ of degree 1. In particular, this implies that we have

$$A \cong k[x_1, \ldots, x_n]/\mathfrak{a},$$

for some ideal $\mathfrak{a} = (f_1, \ldots, f_k)$, where $f_i \in k[x_1, \ldots, x_n]$ are homogeneous elements. The grading of elements in A is inherited from that in $k[x_1, \ldots, x_n]$, i.e. in A we have that

$$ax_1^{i_1} \cdots x_n^{i_n} + \mathfrak{a}$$

is homogeneous of degree $\sum i_k$.

Definition. Let A be a graded ring as above. The **Hilbert function of** A is the function $H_A\colon \mathbb{N} \to \mathbb{N}$ given by $H_A(i) = \dim_k(A_i)$.

Example. Consider $k[x]$. We have $k[x]_n = \{ax^n \mid a \in k\}$ for all n, so $H_{k[x]}(n) = 1$ for all n.

Example. For $k[x_1, \ldots, x_n]$, we have that a basis for $k[x_1, \ldots, x_n]_d$ as a k-module is given by elements

$$x_1^{i_1} \cdots x_n^{i_n}$$

with $i_1 + \cdots + i_n = n$. One can compute the number of such to be

$$\binom{n+d-1}{n-1} = \frac{(n+d-1)(n+d-2)\cdots(d)}{(n-1)!},$$

which gives $H_{k[x_1,\ldots,x_n]}(d) = \binom{n+d-1}{n-1} = \frac{d^{n-1}}{(n-1)!}+$ lower order terms in d.

Example. Let $f \in k[x_1, \ldots, x_n]_i$, and let $A = k[x_1, \ldots, x_n]/(f)$. We have a short exact sequence

$$0 \to k[x_1, \ldots, x_n] \xrightarrow{\cdot f} k[x_1, \ldots, x_n] \to A \to 0,$$

which gives short exact sequences

$$0 \to k[x_1, \ldots, x_n]_{d-i} \xrightarrow{\cdot f} k[x_1, \ldots, x_n]_d \to A_d \to 0,$$

and so

$$\dim_k A_d = \dim_k k[x_1, \ldots, x_n]_d - \dim_k k[x_1, \ldots, x_n]_{d-i}.$$

In particular

$$H_A(d) = \begin{cases} \binom{n+d-1}{n-1} \text{ if } d < i \\ \binom{n+d}{n} - \binom{n+d-i-1}{n-1} \text{ if } d \geq i. \end{cases}$$

Note in particular that for $d \geq i$, we have that $H_A(d)$ is a polynomial in d, of the form

$$\frac{id^{n-1}}{(n-2)!} + \text{ lower order terms}$$

Proposition. *Let A be a graded ring as above. There exists a rational polynomial g and an integer N such that, for $n \geq N$, we have $H_A(n) = f(n)$.*

Lemma. *Let $F\colon \mathbb{N} \to \mathbb{N}$ be a function. Assume that there is an $N \geq 0$ and a rational polynomial g such that*

$$F(n+1) - F(n) = g(n)$$

for all $n \geq N$. Then there exists a polynomial f, with $\deg f = \deg g + 1$, such that $F(n) = f(n)$ for all $n \geq N$.

Proof. Let $d = \deg g$, and write $f = \sum_{i=0}^{d+1} a_i x^i$, $g = \sum_{i=0}^{d} b_i x^i$, with $a_i, b_i \in \mathbb{Q}$. If we know (1) that $f(x+1) - f(x) = g(x)$ and (2) that $f(n) = F(n)$, then it follows by induction that $f(n) = F(n)$ for all $n \geq 0$. The conditions (1) and (2) amount to

$$\text{For } j = 0, \ldots, d, \left(\sum_{i=0}^{d+1} \binom{i}{j} a_i\right) - a_j = b_j$$

$$\sum_{i=0}^{d+1} a_i n^i = F(n).$$

This sytem of equations can be solved uniquely, where the $j = d$ equation determines a_{d+1}, the $j = d-1$ equation determinues a_d and so on, until the equation on the second line determines a_0. □

Proof of proposition. The proof is by induction on the number of generators of A as a k-algebra. Assume A is generated by elements $x_1, \ldots, x_n$, homogeneous of degree 1. If x_1 is not a 0-divisor, we have a short exact sequence

$$0 \to A \xrightarrow{\cdot x_1} A \to A/(x_1) \to 0,$$

which gives, for each i, a short exact sequence

$$0 \to A_i \xrightarrow{\cdot x_1} A_{i+1} \to (A/(x_1))_{i+1} \to 0.$$

We thus have

$$H_A(i+1) = H_A(i) + H_{A/(x_1)}(i+1)$$

or equivalently

$$H_A(i+1) - H_A(i) = H_{A/(x_1)}(i+1).$$

Since $A/(x_1)$ is generated by the elements $x_2, \cdots, x_n$, the induction hypothesis shows that $H_{A/(x_1)}$ is eventually a polynomial, and our lemma shows that the same is then true of H_A.

If x_1 *is* a 0-divisor, the short exact sequence

$$0 \to A/\operatorname{Ann}(x_1) \xrightarrow{\cdot x_1} A \to A/(x_1) \to 0$$

let's us reduce the claim from A to $A/(x_1)$ (handled by induction) and $A/\operatorname{Ann}(x_1)$ (where x_1 is not a 0-divisor, so handled above). □

Definition. We call the polynomial which computes the Hilbert function for large integers the **Hilbert polynomial**.

Theorem. *If A is a graded ring with Hilbert polynomial P_A, satisfying the assumptions above, then* $\deg P_A = \dim A + 1$.

We define the degree of the 0 polynomial to be -1.

Example. For $A = k[x_1, \ldots, x_n]$, we have $P_A(d) = \binom{n+d-1}{n-1}$, so $\deg P_A = n - 1$ and hence $\dim A = n$.

Computing the Hilbert function. There are two main ways to compute Hilbert functions of a graded ring A. The first is to write down an explicit k-basis for A_d, whose cardinality then gives $H_A(d)$, as was done in the case of $A = k[x_1, \ldots, x_n]$ above.

Example. Let $A = k[x, y]/(x^3)$, and let $\bar{x}, \bar{y} \in A$ be the images of x and y, respectively. The k-module A_d has basis

$$\begin{cases} \{1\} \text{ if } d = 0 \\ \{\bar{x}, \bar{y}\} \text{ if } d = 1 \\ \{\bar{x}^2\bar{y}^{d-2}, \bar{x}\bar{y}^{d-1}, \bar{y}^d\} \text{ if } d \geq 2 \end{cases},$$

which gives

$$H_A(d) = \begin{cases} 1 \text{ if } d = 0 \\ 2 \text{ if } d = 1 \\ 3 \text{ if } d \geq 2. \end{cases}$$

Thus the Hilbert polynomial of A is three.

The second method applies if $A = B/(f)$, where $f \in B_i$ and f is not a 0-divisor. Then (defining $B_j = 0$ if $j < 0$) we have an exact sequence of k-modules

$$0 \to B_{d-i} \xrightarrow{\cdot f} B_d \to A_d \to 0,$$

which gives

$$\dim_k A_d = \dim_k B_d - \dim_k B_{d-i},$$

that is

$$H_A(d) = H_B(d) - H_B(d - i),$$

here setting $H_B(j) = 0$ if $j < 0$.

Example. With $B = k[x, y]$ and $A = B/(x^3)$, we get

$$H_A(d) = H_B(d) - H_B(d - 3).$$

Since

$$H_B(d) = \begin{cases} d + 1 \text{ if } d \geq 0 \\ 0 \text{ if } d < 0, \end{cases}$$

this gives

$$H_A(d) = \begin{cases} 1 \text{ if } d = 0 \\ 2 \text{ if } d = 1 \\ 3 \text{ if } d \geq 2. \end{cases}$$

as before.