Suggested problems week 1

(1) Prove that if A is a ring such that $1 = 0$, then A is the 0 ring.
(2) Prove Proposition 1.1. in [AM].
(3) Use the relation between ideals in $\mathbb{Z}$ and $\mathbb{Z}/(n) = \mathbb{Z}_n$ to show that the number of ideals in $\mathbb{Z}_n$ equals the number of positive integers dividing n.
(4) Let k be a field, and prove that the ideal
$$(x, y) = \{g_1 x + g_2 y \mid g_1, g_2 \in k[x, y]\} \subseteq k[x, y]$$
is not a principal ideal.
(5) Let $\mathfrak{a} \subset \mathbb{Z}[x]$ be the set of polynomials such that $f \in \mathfrak{a}$ if and only if $f(0)$ is even. Show that $\mathfrak{a}$ is an ideal, and that it is not a principal ideal.
(6) Prove that a ring A is an integral domain if and only if $A[x]$ is an integral domain.
(7) Convince yourself that for any ring A, we have an isomorphism $(A[x])[y] \cong A[x, y]$
(8) Let A be a ring. Prove that there exists some field k such that there is a surjective homomorphism $\phi\colon A \to k$.
(9) Let A be a ring. Prove that there exists a unique homomorphism $\phi\colon \mathbb{Z} \to A$.
(10) Let $n \geq 1$, and let $\mathbb{Z}^n$ be the ring of n-tuples
$$(a_1, \ldots, a_n) \qquad a_i \in \mathbb{Z},$$
with addition and multiplication defined componentwise. Prove that for any ring A there are exactly n homomorphisms $\phi\colon \mathbb{Z}^n \to A$.
(11) Let A be a ring, and let $a \in A$. Prove that there exists a unique homomorphism $\phi\colon \mathbb{Z}[x] \to A$ such that $\phi(x) = a$. Use this to describe the set of all homomorphisms $\phi\colon \mathbb{Z}[x] \to A$.
(12) Let C be the ring of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$. Prove that for every $x \in \mathbb{R}$, the set
$$\mathfrak{m}_x = \{f \mid f(x) = 0\} \subset C$$
is a maximal ideal.

Let $\mathfrak{a}$ be the set of functions f such that $f(x) = 0$ when x is sufficiently large (i.e. there exists an $y \in \mathbb{R}$ such that $x \geq y \Rightarrow f(x) = 0$)). Prove that $\mathfrak{a}$ is an ideal. Use this to prove that C has a maximal ideal not equal to $\mathfrak{m}_x$ for any $x \in \mathbb{R}$.

From Atiyah–MacDonald chapter 1: 7, 8, 9, 10, 12, 15, 16.
From Ellingsrud: 1.1, 1.10, 2.15, 2.20

Suggested problems week 2

[AM]: 1.1

[E]: 2.2, 2.5, 2.11, 2.14, 2.24, 2.26, 2.28, 2.50 (Here "comaximal" is a synonym for "coprime")

(*) means I suspect a problem is difficult (and you're not likely to miss anything important by skipping it).

(1) Let A be a ring, let $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in A$. Prove that

$$(a_1, \ldots, a_n)(b_1, \ldots, b_m) = (a_1 b_1, a_1 b_2, \ldots, a_n b_m),$$

where the sequence in the rightmost brackets contains every product $a_i b_j$ with $1 \leq i \leq n$ and $1 \leq j \leq m$.

(2) Let A be a ring, and let $a \in A$. Prove that $A[x]/(x-a) \cong A$.

(3) Let A be a ring, let $a, b \in A$. Prove that $A[x]/(x-a, b) \cong A/(b)$.

(4) Let A be a ring and let $\mathfrak{a}, \mathfrak{b}$ be ideals. Prove that $\mathfrak{a}, \mathfrak{b}$ are coprime if and only if there is no maximal ideal $\mathfrak{m}$ with $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{m}$.

(5) Let k be a field, and let $f, g \in k[x]$. Prove that if (f) and (g) are coprime if and only if f and g have no common irreducible factor.[1]

(6) Let k be a field, and $0 \neq f \in k[x]$. Prove that $k[x]/(f)$ is isomorphic to a direct product of rings of the form $k[x]/(g^n)$, where g is irreducible.

(7) Let $(a_1, b_1) \neq (a_2, b_2) \in \mathbb{C}^2$. Prove that the ideals $(x - a_1, y - b_1)$ and $(x - a_2, y - b_2)$ in $\mathbb{C}[x, y]$ are coprime.

(8) Show that the ideals $(x-2)$ and $(2x^2 - 2)$ in $\mathbb{Z}[x]$ are not coprime.

(9) Let $a_1, \ldots, a_n, b \in A$, and let $1 \leq i, j \leq n$ with $i \neq j$. Prove that

$$(a_1, \ldots, a_n) = (a_1, \ldots, a_{j-1}, a_j + b a_i, a_{j+1}, \ldots, a_n)$$

(10) Let $a_1, \ldots, a_n \in A$, let $1 \leq i \leq n$, and let $c \in A$ be a unit. Prove that

$$(a_1, \ldots, a_n) = (a_1, \ldots, a_{i-1}, c a_i, a_{i+1}, \ldots, a_n).$$

(11) (*) Let $a_1, \ldots, a_n \in A$, and let $(b_{ij})_{1 \leq i,j \leq n}$ be an invertible matrix with coefficients in A.[2] Prove that

$$(a_1, \ldots, a_n) = \left(\sum_{j=1}^n b_{1j} a_j, \sum_{j=1}^n b_{2j} a_j, \ldots, \sum_{j=1}^n b_{nj} a_j \right).$$

(12) (*) Let $A = \mathbb{Z}[x]/(x^2 + 1)$, and let $p \in \mathbb{Z}$ be a prime. Let $\phi \colon \mathbb{Z} \to A$ be the natural map, and show that $\phi((p))A \subseteq A$ is a prime ideal if and only if -1 is a quadratic residue modulo p, that is if and only if the equation $x^2 + 1$ admits solutions in $\mathbb{Z}/(p)$. *Hint:* Show that $A/\phi((p))A \cong \mathbb{Z}/(p)[x]/(x^2+1)$ and consider what it means for this ring to be an integral domain.

(13) Let $\mathfrak{a}, \mathfrak{p} \subseteq A$ be ideals with $\mathfrak{p}$ prime such that $\mathfrak{a}$ is not contained in $\mathfrak{p}$. Prove that $(\mathfrak{p} : \mathfrak{a}) = \mathfrak{p}$.

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[1] Recall that in $k[x]$ every ideal is principal, every polynomial can be factored into irreducible polynomials, and $h \in k[x] \setminus \{0\}$ is irreducible if and only if (h) is prime.

[2] This means that there exist $(c_{ij})_{1 \leq i,j \leq n}$ with $c_{ij} \in A$ such that

$$\sum_{k=1}^n b_{ik} c_{kj} = \sum_{k=1}^n c_{ik} b_{kj} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}.$$

(14) Let $p_1, \ldots, p_k$ be distinct prime numbers, let $m = \prod_{i=1}^{k} p_i^{e_i}$ and $n = \prod_{i=1}^{k} p_i^{f_i}$ with $e_i, f_i \geq 0$. Prove that $((m) : (n)) = (r)$ with $r = \prod_{i=1}^{k} p_i^{\max(e_i - f_i, 0)}$.

(15) (*) This exercise proves the implication "$f \in A$ nilpotent" $\Rightarrow f$ is contained in every prime ideal of A.

We will assume that f is not nilpotent, and construct a prime ideal not containing f. Let $N = \{f^n \mid n \geq 0\}$, and let S be the set of ideals $\mathfrak{a}$ of A such that $\mathfrak{a} \cap N = \varnothing$.

(a) Prove that S is non-empty.

(b) Use a Zorn's lemma argument to prove that S has a maximal element under partial ordering.

(c) Let $\mathfrak{p} \in S$ be a maximal element, and let $g \notin \mathfrak{p}$. Prove that we can find $n \geq 0, x \in A, y \in \mathfrak{p}$ such that

$$f^n = y + xg.$$

(d) Prove that $\mathfrak{p}$ is a prime ideal not containing f.