Suggested problems week 1

(1) Prove that if A is a ring such that $1 = 0$, then A is the 0 ring.
(2) Prove Proposition 1.1. in [AM].
(3) Use the relation between ideals in $\mathbb{Z}$ and $\mathbb{Z}/(n) = \mathbb{Z}_n$ to show that the number of ideals in $\mathbb{Z}_n$ equals the number of positive integers dividing n.
(4) Let k be a field, and prove that the ideal
$$(x, y) = \{g_1 x + g_2 y \mid g_1, g_2 \in k[x, y]\} \subseteq k[x, y]$$
is not a principal ideal.
(5) Let $\mathfrak{a} \subset \mathbb{Z}[x]$ be the set of polynomials such that $f \in \mathfrak{a}$ if and only if $f(0)$ is even. Show that $\mathfrak{a}$ is an ideal, and that it is not a principal ideal.
(6) Prove that a ring A is an integral domain if and only if $A[x]$ is an integral domain.
(7) Convince yourself that for any ring A, we have an isomorphism $(A[x])[y] \cong A[x, y]$
(8) Let A be a ring. Prove that there exists some field k such that there is a surjective homomorphism $\phi\colon A \to k$.
(9) Let A be a ring. Prove that there exists a unique homomorphism $\phi\colon \mathbb{Z} \to A$.
(10) Let $n \geq 1$, and let $\mathbb{Z}^n$ be the ring of n-tuples
$$(a_1, \ldots, a_n) \qquad a_i \in \mathbb{Z},$$
with addition and multiplication defined componentwise. Prove that for any ring A there are exactly n homomorphisms $\phi\colon \mathbb{Z}^n \to A$.
(11) Let A be a ring, and let $a \in A$. Prove that there exists a unique homomorphism $\phi\colon \mathbb{Z}[x] \to A$ such that $\phi(x) = a$. Use this to describe the set of all homomorphisms $\phi\colon \mathbb{Z}[x] \to A$.
(12) Let C be the ring of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$. Prove that for every $x \in \mathbb{R}$, the set
$$\mathfrak{m}_x = \{f \mid f(x) = 0\} \subset C$$
is a maximal ideal.

Let $\mathfrak{a}$ be the set of functions f such that $f(x) = 0$ when x is sufficiently large (i.e. there exists an $y \in \mathbb{R}$ such that $x \geq y \Rightarrow f(x) = 0$)). Prove that $\mathfrak{a}$ is an ideal. Use this to prove that C has a maximal ideal not equal to $\mathfrak{m}_x$ for any $x \in \mathbb{R}$.

From Atiyah–MacDonald chapter 1: 7, 8, 9, 10, 12, 15, 16.
From Ellingsrud: 1.1, 1.10, 2.15, 2.20