Suggested problems week 3

From [E]: 4.1, 4.2, 4.3, 4.4, 4.5, 4.11, 4.12, 4.17, 4.18, 4.20 (let p be a positive odd number, not necessarily a prime), 4.21, 4.34 (an ideal $\mathfrak{a}$ is nilpotent if $\mathfrak{a}^n = 0$ for some $n \geq 0$), 5.1., 5.5.

(1) Let G be an abelian group, and let $\phi\colon G \to G$ be a group homomorphism. Prove that there is a unique structure of $\mathbb{Z}[x]$-module on G such that $xg = \phi(g)$ for all $g \in G$.

(2) Let G be an abelian group and $n \geq 1$. Show that G admits a structure of $\mathbb{Z}/(n)$-module if and only if $\overbrace{g + \cdots + g}^{n} = 0$ for all $g \in G$. Show that if this condition holds, then the structure of $\mathbb{Z}/(n)$-module on G is unique.

(3) Let A be an integral domain and $0 \neq a \in A$. Show that A and the ideal (a) are isomorphic as A-modules.

(4) Let M be an A-module. Show that the map $F\colon \mathrm{Hom}_A(A, M) \to M$ given by
$$F(\phi) = \phi(1)$$
is an isomorphism of A-modules.

(5) Let A be a ring and $A[[x]]$ be the ring of power series with coefficients in A. Think of $A[[x]]$ as an A-module via $a(\sum b_i x^i) = \sum ab_i x^i$. Prove that
$$A[[x]] \cong \prod_{i \in \mathbb{N}} A$$
as an A-module.

(6) Let $M_1, \ldots, M_n$ and N be A-modules. Prove that $\mathrm{Hom}(M_1 \oplus \cdot \oplus M_n, N)$ is isomorphic as an A-module to $\mathrm{Hom}(M_1, N) \oplus \cdots \oplus \mathrm{Hom}(M_n, N)$.

(7) Let $\{M_i\}_{i \in S}$ and N be A-modules. Prove that
$$\mathrm{Hom}(\bigoplus_{i \in S} M_i, N) \cong \prod_{i \in S} \mathrm{Hom}(M_i, N)$$
and
$$\mathrm{Hom}(N, \prod_{i \in S} M_i) \cong \prod_{i \in S} \mathrm{Hom}(N, M_i)$$
as A-modules.

(8) Let A be a ring, let $M = A \oplus A$, and let $\phi\colon M \to M$ be the homomorphism given by $\phi(x, y) = (0, x)$. Prove that the sequence
$$\cdots \xrightarrow{\phi} M \xrightarrow{\phi} M \xrightarrow{\phi} M \xrightarrow{\phi} \cdots$$
is exact.

(9) Let k be a field, let $n \geq 3$, and let
$$0 \to V_1 \to \cdots \to V_n \to 0$$
be an exact sequence of finite-dimensional vector spaces. Prove that the homomorphism $V_{n-2} \to V_{n-1}$ is surjective if and only if $\sum_{i=1}^{n-1} (-1)^i \dim V_i = 0$.

(10) Let $A \subset \mathbb{R}(x)$ be the ring of rational functions defined at 0 as in the lecture notes, with maximal ideal $\mathfrak{m} \subset A$. Thinking of $\mathbb{R}(x)$ as an A-module, prove that $\mathbb{R}(x) = \mathfrak{m}\mathbb{R}(x)$. Why does this not contradict Nakayama's lemma?

(11) Let A be a local ring with maximal ideal $\mathfrak{m}$, let $\phi\colon N \to M$ be a homomorphism of A-modules with M finitely generated. Use the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}N & \longrightarrow & N & \longrightarrow & N/\mathfrak{m}N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{m}M & \longrightarrow & M & \longrightarrow & M/\mathfrak{m}M & \longrightarrow & 0, \end{array},$$

to show that if $N/\mathfrak{m}N \to M/\mathfrak{m}M$ is an isomorphism, then $\mathfrak{m}M \subseteq \phi(\mathfrak{m}N)$.