Suggested problems week 6

From [AM]: 3.14
From [E]: 7.14, 7.18, 7.21

(1) Let $\mathfrak{a} \subseteq A$ be an ideal, and let $\mathfrak{p}$ be a prime ideal such that $\mathfrak{a}$ is not contained in $\mathfrak{p}$. Prove that $(A/\mathfrak{a})_\mathfrak{p} = 0$ and that $\mathfrak{a}_\mathfrak{p} \cong A_\mathfrak{p}$ as $A_\mathfrak{p}$-modules.

(2) Let M be an A-module, and let $f \in A$ be such that $fm = 0$ for all $m \in M$. Prove that $M_P = 0$ for all prime ideals $P \subset A$ with $f \notin P$.

(3) Let M be a *finitely generated* A-module, and let $P \subset A$ be a prime ideal such that $M_P = 0$. Prove that there exists an $f \in A \setminus P$ such that $fm = 0$ for all $m \in M$.

(4) Show that the condition "finitely generated" in the previous problem is necessary as follows. Let M be the $\mathbb{Z}$-module

$$M = \bigoplus_{n \geq 1} \mathbb{Z}/(n).$$

Prove that $M_{(0)} = 0$ and that there is no $f \in \mathbb{Z}$ such that $fm = 0$ for all $m \in M$.

(5) Prove that the functor $M \mapsto S^{-1}M$ does not commute with infinite intersections of submodules; that is, give an example of a ring A, a multiplicatively closed subset $S \subseteq A$, an A-module M, and an infinite set of submodules $\{M_i\}_{i \in T}$ of M such that

$$\bigcap_{i \in T} S^{-1}M_i \neq S^{-1}(\bigcap_{i \in T} M_i).$$

Hint: There is an example with $A = \mathbb{Z}, S = \mathbb{Z} \setminus \{0\}$.