

MAT4200

Mandatory assignment 1 of 2

Submission deadline

Thursday 22nd of October 2026, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with L^AT_EX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

You are encouraged to work with each other on the problems but you must write up (by hand or in Latex) your own solutions. If you collaborate on problems, acknowledge your collaborators at the beginning of the solution.

Part 2 of the Obligatory assignment will be an informal meeting and discussion surrounding your written solutions depending on their quality.

Problem 1. Let R be a ring in which every element satisfies $x^n = x$ for some $n > 1$ depending on x . Prove that every prime ideal of R is maximal.

Problem 2. Let G be an abelian group, Mod_R denote the set of R -modules, and let $\mathcal{C}$ be a class of modules in Mod_R . An **additive function** is a function $\chi : \mathcal{C} \rightarrow G$ such that for every short exact sequence of modules in $\mathcal{C}$

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

the function satisfies

$$\chi(M) = \chi(M') + \chi(M'').$$

1. Suppose k is a field and $n \geq 3$. Let

$$0 \rightarrow V_1 \rightarrow \cdots \rightarrow V_n \rightarrow 0$$

be an exact sequence of finite-dimensional vector spaces. Prove that the homomorphism $V_{n-2} \rightarrow V_{n-1}$ is surjective if and only if $\sum_{i=1}^{n-1} (-1)^i \dim V_i = 0$. Hint: $\dim$ is an additive function on the class of finite dimensional vector spaces. How does an additive function behave on long exact sequences?

2. Let $\mathcal{C}$ be the class of finite $\mathbb{Z}$ -modules and G the multiplicative group $\mathbb{Q}_{>}$. Show that the function $\chi(M) = |M|$ is additive in the above sense (i.e. meaning that $\prod_{i=1}^n |M_i|^{-1^i} = 1$).

Problem 3. A **free resolution** of a module M is an exact sequence of the form,

$$\cdots F_3 \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0,$$

where F_i are free R -modules for all i . A free resolution is finite if there is some N such that $F_n = 0$ for all $n \geq N$ and is of finite rank if all F_i are finite rank. Note the 0 module has rank 0.

1. Find a free resolution of the $\mathbb{Z}$ -module $M = \mathbb{Z}/(n)$ for all n .
2. Find a finite rank resolution of the $\mathbb{Z}/(4)$ -module $M = \mathbb{Z}/(2)$. Show that there is no finite free resolution of finite rank. **Hint:** use the additive function in Part 2 of Problem 2.
3. Let $R = \mathbb{Q}[x, y]$ and let $M = R/I$ where $I = (x^n, y^m)$. Construct a free resolution where $F_i = 0$ for all $i \geq 3$.

Problem 4. Let $f : R \rightarrow S$ be a ring homomorphism. Prove that if M is a flat R -module, then $S \otimes_R M$ is a flat S -module.

Using the above statement, prove:

1. if M is a flat R -module, then the localisation $M_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$ -module for any prime ideal $\mathfrak{p}$.
2. if M is a flat R module, then M/IM is a flat R/I -module.

Problem 5. An R -module is **torsion free** if the map $\phi_r : M \rightarrow M$ given by $\phi_r(m) = rm$ is injective for all $r \neq 0$.

1. Prove that the property of being torsion free is local, i.e. that M is torsion free if and only if M_P is torsion free for all prime ideals $P \subset R$.
2. Prove that if R is an integral domain and M is a free R -module then M is torsion free.
3. Give an example when R is not an integral domain and statement (2) fails.

Problem 6. Let M be an R -module. Prove that if every submodule of M is finitely generated, then for any increasing sequence of submodules of M

$$M_1 \subseteq M_2 \subseteq M_3 \dots$$

there exists a n such that $M_k = M_n$ for all $k \geq n$. **Hint:** consider the submodule of M which is the infinite union of the members of the sequence.