

MAT4200 - 2026 EXERCISES WEEK 1

Warm ups

- (1) Let R be a ring. Using the ring axioms prove that $0a = 0$ for all $a \in R$ where 0 is the additive identity element. Prove also that $(-a)(-b) = ab$ and $(-a)b = -(ab)$.
- (2) Prove that if R is a ring such that $1 = 0$, then R is the 0 ring.
- (3) Prove that if $\phi : S \rightarrow R$ satisfies $\phi(a + b) = \phi(a) + \phi(b)$, $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in S$ and $\phi(1_S) = 1_R$, then it is also a homomorphism of the additive groups $(S, +) \rightarrow (R, +)$.
- (4) Use the relation between ideals in $\mathbb{Z}$ and $\mathbb{Z}/(n) = \mathbb{Z}_n$ to show that the number of ideals in $\mathbb{Z}_n$ equals the number of positive integers dividing n .
- (5) Let R be a ring. Prove that there exists some field k such that there is a surjective homomorphism $\phi : R \rightarrow k$.

Workout

- (1) Let R be a non-zero ring. Show that the ideal
$$(x, y) = \{g_1x + g_2y \mid g_1, g_2 \in R[x, y]\} \subseteq R[x, y]$$
is not a principal ideal.
- (2) Let $I \subset \mathbb{Z}[x]$ be the set of polynomials such that $f \in I$ if and only if $f(0)$ is even. Show that I is an ideal by exhibiting it as a kernel of a map of rings. Is the ideal principal?
- (3) Prove that a ring R is an integral domain if and only if $R[x]$ is an integral domain.
- (4) Convince yourself that for any ring R , we have an isomorphism $(R[x])[y] \cong R[x, y]$.
- (5) Let R be a ring. Prove that there exists a unique homomorphism $\phi : \mathbb{Z} \rightarrow R$.
- (6) Let R be a ring, and let $a \in R$. Prove that there exists a unique homomorphism $\phi : \mathbb{Z}[x] \rightarrow R$ such that $\phi(x) = a$. Use this to describe the set of all homomorphisms $\phi : \mathbb{Z}[x] \rightarrow R$.
- (7) Let C be the ring of continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$. Prove that for every $x \in \mathbb{R}$, the set

$$\mathfrak{m}_x = \{f \mid f(x) = 0\} \subset C$$

is a maximal ideal.

Let I be the set of functions f such that $f(x) = 0$ when x is sufficiently large (i.e. there exists an $y \in \mathbb{R}$ such that $x \geq y \Rightarrow f(x) = 0$). Prove that I is an ideal. Use this to prove that C has a maximal ideal not equal to $\mathfrak{m}_x$ for any $x \in \mathbb{R}$.

From Atiyah–MacDonald chapter 1: 1, 2, 7, 15

From Ellingsrud: 1.1, 1.2, 1.8, 1.10, 2.15