MAT4200 - 2026 EXERCISES WEEK 10

Warm ups

(1) Let R be a ring. Is $R[x]$ Noetherian? Is $R[x]$ Artinian?
(2) What is $\dim(\mathbb{Z})$? What is $\dim(k)$ for k a field?
(3) Is $\mathbb{Z}$ Artinian? Is k Artinian?

Suggested problems week 10

(1) Let P be a prime ideal of a ring R show that

$$\dim(R) \geq \dim(R/P) + \dim(R_P),$$

where dim denotes the Krull dimension. This holds even if some dimensions are not finite.
(2) Study Proposition 8.4 of Atiyah-Macdonald. Use this proposition to show that there exists a number e such that $\mathfrak{m}_1^e \dots \mathfrak{m}_n^e = 0$ where $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ are the maximal ideals of an Artinian ring R.
(3) Let R be a local Noetherian ring with maximal ideal $\mathfrak{m}$. Prove that R is Artinian if and only if $\mathfrak{m}$ is nilpotent
(4) Prove that an Artinian ring R has a finite number of prime ideals. Hence $\mathrm{Spec}(R)$ is a finite topology.

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Extra curricular

(1) Let R be Noetherian. Prove the following are equivalent.
- R is Artinian
- $\mathrm{Spec}(R)$ is a finite and discrete topology. (recall discrete topology means every subset is an open set).
- $\mathrm{Spec}(R)$ is a discrete topology.